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Statistical Physics with Continuous Symmetries

VI. The Bose (lattice) Gas and Stochastic Representations

The Bose Lattice Gas:

N Bose-particles (bosons) on the lattice $\Lambda \subset \mathbb{Z}^d$:

$$\mathcal{H}_{\Lambda,N} = \{f : \Lambda^N \to \mathbb{C} : \forall \pi \in S_N : \; f(\pi \underline{x}) = f(\underline{x}), \; (f,g) := \sum_{\underline{x}} \overline{f(\underline{x})} g(\underline{x})\}$$

$$H_{\Lambda,N} = \underbrace{\sum_{1 \leq j \leq N} -\Delta^{(j)}}_{\text{kinetic energy}} + \underbrace{\sum_{1 \leq i < j \leq N} V(x_i - x_j)}_{\text{interaction energy}} + \underbrace{h\,N\,I}_{\text{dummy}}$$

HW: Compute $\dim(\mathcal{H}_{\Lambda,N})$.

Unspecified number of bosons on Λ:

$$\mathcal{H}_\Lambda = \bigoplus_{N=0}^{\infty} \mathcal{H}_{\Lambda,N}, \qquad\qquad H_\Lambda|_{\mathcal{H}_{\Lambda,N}} = H_{\Lambda,N}$$

The role of h: Control the number of particles in the system.

Condensation in the Free Bose Gas $V = 0$:

[SN Bose (1924)], [A Einstein (1924)]:

If $V = 0$ then the Hamiltonian $H_{\Lambda,N}$ is easily diagonalizable. Let

$$\varphi_{\Lambda,p}(x) := |\Lambda|^{-1/2}\, e^{ip\cdot x}, \qquad p \in \Lambda^*$$

be the Fourier o-n basis in $\ell^2(\Lambda)$,

$$[N|\Lambda^*] := \{\underline{n} : \Lambda^* \to \mathbb{N}, \quad \text{such that } \sum_{p\in\Lambda^*} n(p) = N\}$$

and, for $\underline{n} \in [N|\Lambda^*]$, $(p_r(\underline{n}))_{1\le r\le N}$ be a (canonical) listing of the p-s in $\underline{n}$, with multiplicities. Then

$$\psi_{\Lambda,N,\underline{n}}(x_1,\ldots,x_N) := (N!)^{-1/2} \sum_{\pi\in S_N} \prod_{r=1}^{N} \varphi_{\Lambda,p_r(\underline{n})}(x_{\pi(r)}), \qquad \underline{n} \in [n|\Lambda^*],$$

form an orthonormal basis in $\mathcal{H}_{\Lambda,N}$ (HW).

These are exactly the eigenvectors of the free Hamiltonian (HW).

$$H_{\Lambda,N}\psi_{\Lambda,N,\underline{n}} = \epsilon_{\Lambda,N,\underline{n}}\psi_{\Lambda,N,\underline{n}}$$

with eigenvalues

$$\epsilon_{\Lambda,N,\underline{n}} = \sum_{p\in\Lambda^*} D(p)n(p)$$

We also define the *occupation number operators* for the one-particle states $\left(\varphi_{\Lambda,p}\right)_{p\in\Lambda^*}$, in the diagonal form:

$$N_{\Lambda,N}(p)\psi_{\Lambda,N,\underline{n}} = \underline{n}(p)\psi_{\Lambda,N,\underline{n}}, \qquad \underline{n} \in [N|\Lambda^*]$$

Note that

$$\sum_{p\in\Lambda^*} N_{\Lambda,N}(p) = NI$$

Putting these together we write

$$H_{\Lambda,N} = \sum_{p \in \Lambda^*} D(p) N_{\Lambda,N}(p)$$

Thus, we can compute explicitly whatever we please.

Let $\beta < \infty$ be fixed, and compute the occupation density of the one-particle ground state $|\Lambda|^{-1} \left\langle N_{\Lambda,N}(0) \right\rangle_{\Lambda,N,\beta}$, or rather

$$|\Lambda|^{-1} \sum_{p \in \Lambda \setminus \{0\}} \left\langle N_{\Lambda,N}(p) \right\rangle_{\Lambda,N,\beta} = |\Lambda|^{-1} N - |\Lambda|^{-1} \left\langle N_{\Lambda,N}(0) \right\rangle_{\Lambda,N,\beta}$$

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$$|\Lambda|^{-1} \sum_{p\in\Lambda\backslash\{0\}} \left\langle N_{\Lambda,N}(p)\right\rangle_{\Lambda,N,\beta} =$$

$$= |\Lambda|^{-1} \sum_{p\in\Lambda\backslash\{0\}} \frac{\sum_{\underline{n}\in[N|\Lambda^*]} n(p) \prod_{q\in\Lambda^*\backslash\{0\}} e^{-\beta D(q)n(q)}}{\sum_{\underline{n}\in[N|\Lambda^*]} \prod_{q\in\Lambda^*\backslash\{0\}} e^{-\beta D(q)n(q)}}$$

$$= |\Lambda|^{-1} \sum_{p\in\Lambda\backslash\{0\}} \frac{\sum_{M\leq N} \sum_{\underline{n}\in[M|\Lambda^*\backslash\{0\}]} n(p) \prod_{q\in\Lambda^*\backslash\{0\}} e^{-\beta D(q)n(q)}}{\sum_{M\leq N} \sum_{\underline{n}\in[M|\Lambda^*\backslash\{0\}]} \prod_{q\in\Lambda^*\backslash\{0\}} e^{-\beta D(q)n(q)}}$$

$$= \mathbf{E}\Big(|\Lambda|^{-1} \sum_{p\in\Lambda\backslash\{0\}} \xi_{\Lambda^*,p} \,\Big|\, |\Lambda|^{-1} \sum_{p\in\Lambda\backslash\{0\}} \xi_{\Lambda^*,p} \leq |\Lambda|^{-1} N \Big) \qquad (1)$$

where $\left(\xi_{\Lambda^*,p}\right)_{p\in\Lambda^*\backslash\{0\}}$ are independent random variables with geometric distributions

$$\mathbf{P}\Big(\xi_{\Lambda^*,p} = k\Big) = (1 - e^{-\beta D(p)})e^{-\beta D(p)k} \sim \mathtt{GEOM}(e^{-\beta D(p)})$$

$$\mathbf{E}\Big(\xi_{\Lambda^*,p}\Big) = \frac{e^{-\beta D(p)}}{1 - e^{-\beta D(p)}} \qquad \mathbf{Var}\Big(\xi_{\Lambda^*,p}\Big) = \frac{e^{-\beta D(p)}}{(1 - e^{-\beta D(p)})^2}$$

Computing the conditional expectation on the rhs of (1), in the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^d$, $N \to \infty$, $N/|\Lambda| \to \varrho \in (0, \infty)$, becomes a well-posed *large deviation problem* of probability theory. Let

$$\begin{aligned}\varrho^* &:= \lim_{\Lambda \nearrow \mathbb{Z}^d} |\Lambda|^{-1} \sum_{p \in \Lambda \setminus \{0\}} \mathbf{E}\Big(\xi_{\Lambda^*,p}\Big) \\ &= \int_{[-\pi,\pi]^d} \frac{e^{-\beta D(p)}}{1 - e^{-\beta D(p)}} dp \begin{cases} = \infty & d = 1, 2 \\ < \infty & d \geq 3 \end{cases}\end{aligned}$$

Then, in the thermodynamic limit

$$\lim \mathbf{E}\Big(|\Lambda|^{-1} \sum_{p\in\Lambda\setminus\{0\}} \xi_{\Lambda^*,p} \Big| \, |\Lambda|^{-1} \sum_{p\in\Lambda\setminus\{0\}} \xi_{\Lambda^*,p} \le \varrho \Big) = \begin{cases} \varrho & \text{if} \quad \varrho \le \varrho^* \\ \varrho^* & \text{if} \quad \varrho > \varrho^* \end{cases} \tag{2}$$

and thus

$$\varrho_{\text{cond}} := \lim \frac{\langle \#\text{particles in single particle ground state} \rangle_{\Lambda,\beta}}{|\Lambda|} = (\varrho - \varrho^*)_+$$

Topics for essay for a probabilist student: Prove BEC with full mathematical rigour.

The limit (2) follows from well established, though not completely trivial probabilistic arguments:

- $\varrho < \varrho^*$, $d \geq 1$: A large deviation estimate in the spirit of Cramér's upper bound
- $\varrho \geq \varrho^*$, $d \geq 4$: Chebyshev's inequality.
- $\varrho \geq \varrho^*$, $d = 3$: Separate the sum as

$$\sum_{p \in \Lambda \setminus \{0\}} \xi_{\Lambda^*,p} = \sum_{\substack{p \in \Lambda \setminus \{0\} \\ |p| < \varepsilon}} \xi_{\Lambda^*,p} + \sum_{\substack{p \in \Lambda \setminus \{0\} \\ |p| \geq \varepsilon}} \xi_{\Lambda^*,p}$$

then apply Markov's inequality to the first and Chebyshev's inequality to the second part.

Still BEC. Fock space and second quantization.
Unitary equivalent reformulation of the same setting (HW).

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \ell^2(\mathbb{N}) = \bigoplus_{N=0}^{\infty} \ell^2_{\mathsf{symm}}(\Lambda^N) = \bigoplus_{N=0}^{\infty} \left(\bigotimes_{j=1}^{N} \ell^2(\Lambda) \right)_{\mathsf{symm}}$$

$$H_\Lambda = - \sum_{x\sim y\in\Lambda} \mathfrak{b}^\dagger(x)\mathfrak{b}(y) + \frac{1}{2} \sum_{x,y\in\Lambda} V(y-x)\{\mathfrak{n}(x)\mathfrak{n}(y)\}_{\mathsf{p}} + h \sum_x \mathfrak{n}(x)$$

- $\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{n}$ are *bosonic creation, annihilation, number* operators acting on $\ell^2(\mathbb{N})$:

$$[\mathfrak{b}, \mathfrak{b}^\dagger] = I, \qquad \mathfrak{n} = \mathfrak{b}^\dagger \mathfrak{b}, \qquad [\mathfrak{n}, \mathfrak{b}^\dagger] = \mathfrak{b}^\dagger, \qquad [\mathfrak{n}, \mathfrak{b}] = -\mathfrak{b},$$

  Their canonical matrix representation is ($n, m \in \mathbb{N}$)

$$\mathfrak{b}^\dagger_{n,m} = \sqrt{n}\delta_{n,m+1} \qquad \mathfrak{b}_{n,m} = \sqrt{n+1}\delta_{n,m-1} \qquad \mathfrak{n}_{n,m} = n\delta_{n,m}$$

- Notational convention (as before): $\mathfrak{a}(x) := I\otimes \cdot\cdot \otimes I \otimes \mathfrak{a} \otimes I \otimes \cdot\cdot \otimes I$
- We count pairs: $\{\mathfrak{n}(x)\mathfrak{n}(y)\}_{\mathsf{p}} := \mathfrak{n}(x)\mathfrak{n}(y) - \delta(x-y)\mathfrak{n}(x)$

Note the formal similarities with the the XXZ Hamiltonian!

In particular: Let the total number operator be

$$N_\Lambda := \sum_{x\in\Lambda} \mathfrak{n}(x), \qquad N_\Lambda|_{\mathcal{H}_{\Lambda,N}} = N\, I|_{\mathcal{H}_{\Lambda,N}}$$

Then, obviously (HW)

$$[N_\Lambda, H_\Lambda] = 0$$

and, thus, the Hamiltonian has the $U(1)$ internal symmetry

$$e^{i\theta N_\Lambda} H_\Lambda e^{-i\theta N_\Lambda} = H_\Lambda$$

It is the case that the BEC *exactly* corresponds to the LRO breaking the $U(1)$ symmetry:

$$\varrho_{\mathrm{cond}} = \lim_{\Lambda\nearrow\mathbb{Z}^d} |\Lambda|^{-2} \sum_{x,y\in\Lambda} \left\langle \mathfrak{b}^\dagger(y)\mathfrak{b}(x)\right\rangle_\Lambda = \lim_{\Lambda\nearrow\mathbb{Z}^d} |\Lambda|^{-1} \sum_{x\in\Lambda} \left\langle \mathfrak{b}^\dagger(0)\mathfrak{b}(x)\right\rangle_\Lambda$$

Condensation in Interacting BG, $V \neq 0$, remains a mystery.

[F London (1938)], [RP Feynman (1953)] :

Superfluidity of liquid $\mathrm{He}^4 = \mathrm{BEC}$. Hence BEC is a major issue in quantum statistics and condensed matter physics.

It is not even clear, however, how to define the "condensate" for interacting bosons.

[O Penrose, L Onsager (1956)]: **ODLRO**

$$\varrho_{\mathrm{cond}} = \lim_{\Lambda \nearrow \mathbb{Z}^d} |\Lambda|^{-1} \Big\langle \underbrace{\underbrace{\Big(|\Lambda|^{-1/2} \sum_{x\in\Lambda} \mathfrak{b}^\dagger(x) \Big)}_{\mathfrak{B}^\dagger} \underbrace{\Big(|\Lambda|^{-1/2} \sum_{y\in\Lambda} \mathfrak{b}(y) \Big)}_{\mathfrak{B}}}_{\mathfrak{N}} \Big\rangle_{\Lambda,\beta}$$

$$= \lim_{\Lambda \nearrow \mathbb{Z}^d} |\Lambda|^{-1} \sum_{x\in\Lambda} \left\langle \mathfrak{b}^\dagger(0)\mathfrak{b}(x) \right\rangle_{\Lambda,\beta}$$

Note: Formal similarity with the LRO $r(\beta)$ in Heisenberg's XXZ model! Actually: **the same type of phase transition**.

The Feynman-Kac Formula

[RP Feynman (1942, PhD)], [M Kac (1949)]

Let $|i\rangle$, $i \in \mathcal{J}$, be a *natural* o-n basis in $\mathcal{H}$, in which $H = -G + V$,

$$G_{i,j} = (1 - \delta_{i,j})|G_{i,j}| - \delta_{i,j} \sum_{k \neq i} G_{i,k}, \qquad V_{i,j} = V(i)\delta_{i,j}$$

Then (under conditions ...) G is the *infinitesimal generator* of the Markov process $t \mapsto \eta(t)$ on the state space $\mathcal{J}$, w jump rates

$$\mathbf{P}\Big(\eta(t + dt) = j \Big| \eta(t) = i\Big) = G_{i,j} dt + o(dt)$$

The following identity holds:

$$\langle i | e^{-tH} | j \rangle = \mathbf{E}\Big(e^{-\int_0^t V(\eta(s)) ds} \mathbb{1}_{\{\eta(t) = j\}} \Big| \eta(0) = i \Big) \qquad \textbf{(FK)}$$

Proof: Both sides are equal to the (unique) sln $f : \mathbb{R}_+ \times \mathcal{J} \to \mathbb{R}$ of the parabolic PDE (HW)

$$\partial_t f = (G - V) f, \qquad f(0, j) = \delta_{i,j}.$$

Remarks to FK:

- **To derive (FK)** write

$$\mathbf{E}\Big(e^{-\int_0^t V(\eta(s))ds}\mathbb{1}_{\{\eta(t)=j\}}\Big|\eta(0)=i\Big) =$$
$$\mathbf{E}\Big(e^{-\int_0^\varepsilon V(\eta(s))ds}\,\mathbf{E}\Big(e^{-\int_\varepsilon^t V(\eta(s))ds}\mathbb{1}_{\{\eta(t)=j\}}\Big|\eta(\varepsilon)\Big)\Big|\,\eta(0)=i\Big)$$

  and use Markov property. Discrete-space is technically easy. The true technical difficulties come with continuous space:

$$\langle f|e^{-t(-\Delta+V)}|g\rangle = \int_{\mathbb{R}^d}\mathbf{E}_x\Big(e^{-\int_0^t V(B(s))ds}g(B(t))\Big)f(x)dx$$

- **Feynman's dream:** Express $\langle f|e^{\sqrt{-1}(-\Delta+V)t}|g\rangle$ as path integral.
- **A quote** from Mark Kac: *Enigmas of Chance* (autobiography):

"It is only fair to say that I had Wiener's shoulders to stand on. Feynman, as in everything else he has done, stood on his own, a trick of intellectual contortion that he alone is capable of."

FK applied to Bose gas: [RP Feynman (1953)]

N bosons in $\Lambda \subset \mathbb{Z}^d$, with pair interaction $V(y-x)$:

$$\mathcal{H}_{\Lambda,N} = \ell^2(\Lambda^N)_{\text{symm}}, \qquad H_{\Lambda,N} = -\sum_i \Delta_i + \sum_{i<j} V(x_i - x_j).$$

[CPF] $$Q_{\Lambda,N}(\beta) := \mathrm{Tr}\Big(e^{-\beta H_{\Lambda,N}}\Big) =$$

$$\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\sum_{x_1,\dots,x_N\in\Lambda} \mathbf{E}_{x_1,\dots,x_N}\Big(e^{-\int_0^\beta \sum_{i<j} V(X_i(s)-X_j(s))ds}\mathbb{1}_{\{X_j(\beta)=x_{\sigma(j)}\}}\Big)$$

[GCPF] $$\Xi_\Lambda(\beta,h) = \sum_{N=0}^{\infty} e^{\beta h N} Q_{\Lambda,N}(\beta)$$

where $X_j(t)$, $1 \le j \le N$ are independent rw-s on Λ.

Define on $\mathcal{S}_N$ the probability measure (next slide)

$$\mathbf{P}_{\Lambda,N,\beta}\big(\sigma\big) = \frac{1}{Q_{\Lambda,N}(\beta)N!}\times$$

$$\sum_{x_1,\dots,x_N\in\Lambda} \mathbf{E}_{x_1,\dots,x_N}\Big(e^{-\int_0^\beta \sum_{i<j} V(X_i(s)-X_j(s))ds}\mathbb{1}_{\{X_j(\beta)=x_{\sigma(j)}\}}\Big)$$

Note: $(\forall \tau \in \mathcal{S}_n)$: $\mathbf{P}_{\Lambda,N,\beta}\big(\tau\,\sigma\,\tau^{-1}\big) = \mathbf{P}_{\Lambda,N,\beta}\big(\sigma\big)$

[RP Feynman (1953)]: In the limit $N \to \infty$, $\Lambda \nearrow \mathbb{Z}^d$, $N/|\Lambda| \to \varrho$, relate *macroscopic size cycles* of $\sigma \sim \mathbf{P}_{\Lambda,N,\beta}$ to BEC.

[A Sütő (1993, 2002)]: For the *free Bose-gas* ($V \equiv 0$) and some *mean field approximations*:

$$\Big\{\varrho_{\text{cond}} > 0\Big\} \Leftrightarrow \Big\{\underline{\lim}\,\mathbf{P}_{\Lambda,N,\beta}\Big(\text{longest cycles of } \sigma \asymp N\Big) > 0\Big\}$$

Condensation of *interacting bosons* ($V \neq 0$) **remains wide open**.

The $s = 1/2$ quantum-XXZ as hard core Bose gas

[TD Holstein, H Primakoff (1940)]:

The Pauli matrices:

$$S_+ := S_1 + iS_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_- := S_1 - iS_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad S_+ S_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$S_+^2 = S_-^2 = 0, \qquad [S_-, S_+] = I - 2S_+S_-,$$

These are exactly the CCR for *bosons with hard core repulsion*.

The Hamiltonian: $\mathcal{H}_\Lambda = (\mathbb{C}^2)^{\otimes \Lambda}$,

$$\begin{aligned} H_\Lambda &= -\sum_{x\sim y} \Big(S_+(x)S_-(y) + uS_3(x)S_3(y) \Big) - h\sum_x S_3(x) \\ &= -\sum_{x\sim y} \mathfrak{b}^\dagger(x)\mathfrak{b}(y) + \frac{1}{2}\sum_{x,y} V(x-y)\{\mathfrak{n}(x)\mathfrak{n}(y)\}_\mathrm{p} - h\sum_x \mathfrak{n}(x) \end{aligned}$$

$$V(x-y) = +\infty \mathbb{1}_{\{x=y\}} - 2u\mathbb{1}_{\{x\sim y\}}$$

The rest based on [BT (1993)] and some follow-up. Ingredients:

- $X_{j,t}$, $1 \le j \le N$, indep. cont. time rws on Λ, w/ j.r. 1/edge.
- $\tau := \inf\{t \ge 0 : X_{i,t} = X_{j,t},\ 1 \le i < j \le N\}$ = first coll. time.
- $\underline{x}_N := \{x_1, \ldots, x_N\} \in \binom{\Lambda}{N}$, where $x_i \ne x_j \in \Lambda$, $1 \le i < j \le N$.
- $\mathcal{B}(A) := \#\{(x,y) \in A \times A : |x-y| = 1\}$ (double count!)
- **Symmetric Simple Exclusion Proc.**: $t \mapsto \eta_t \in \binom{\Lambda}{N}$ - explain.
- **Random Transposition Process**: $t \mapsto \xi_t \in \mathcal{S}_\Lambda$ - explain.

Note: $\forall A \in \binom{\Lambda}{N}$ and $\forall \beta < 0$:

$$\left(\underline{X}_{N,s} : 0 \le s \le \beta < \tau \,\middle|\, \underline{X}_{N,0} = A\right) \stackrel{\text{law}}{=} \left(\eta_s : 0 \le s \le \beta < \tilde{\tau} \,\middle|\, \eta_0 = A\right)$$

where

$$\mathbf{P}_\Lambda\left(\tilde{\tau} > \beta \,\middle|\, \eta(s),\ 0 \le s \le \beta\right) = e^{-\int_0^\beta \mathcal{B}(\eta_s)ds}$$

The CPF in terms of Simple Exclusion:

$$Q_{\Lambda,N}(\beta) = \sum_{A \in \binom{\Lambda}{N}} \mathbf{E}_\Lambda\Big(e^{u\int_0^\beta \mathcal{B}(\underline{X}_{N,s})ds}\mathbb{1}_{\{\tau>\beta\}}\mathbb{1}_{\{\underline{X}_{N,\beta}=A\}}\Big|\,\underline{X}_{N,0}=A\Big)$$

$$= \sum_{A \in \binom{\Lambda}{N}} \mathbf{E}_\Lambda\Big(e^{u\int_0^\beta \mathcal{B}(\eta_s)ds}\mathbb{1}_{\{\widetilde{\tau}>\beta\}}\mathbb{1}_{\{\eta_\beta=A\}}\Big|\,\eta_0=A\Big)$$

$$= \sum_{A \in \binom{\Lambda}{N}} \mathbf{E}_\Lambda\Big(e^{(u-1)\int_0^\beta \mathcal{B}(\eta_s)ds}\mathbb{1}_{\{\eta_\beta=A\}}\Big|\,\eta_0=A\Big)$$

$$\stackrel{u=1}{=} \sum_{A \in \binom{\Lambda}{N}} \mathbf{P}_\Lambda\Big(\eta_\beta=A\,\Big|\,\eta_0=A\Big)$$

From now on $u=1$, the *isotorpic ferromagnet* case.

The GCPF in terms of Random Transpositions:

$$\Xi_\Lambda(\beta, 2h) = \sum_{N=0}^{\infty} e^{2\beta h N} Q_\Lambda(\beta, N)(\beta)$$

$$= \sum_{A \subseteq \Lambda} e^{2\beta h |A|} \mathbf{P}_\Lambda\Big(\eta_\beta = A \,\Big|\, \eta_0 = A\Big)$$

$$= (1 + e^{2\beta h})^{|\Lambda|} \sum_{A \subseteq \Lambda} \Big(\underbrace{\frac{1}{1 + e^{2\beta h}}}_{1-p}\Big)^{|\Lambda \setminus A|} \Big(\underbrace{\frac{e^{2\beta h}}{1 + e^{2\beta h}}}_{p}\Big)^{|A|} \mathbf{P}_\Lambda\Big(\xi_\beta(A) = A\Big)$$

$$\overset{\text{☺☺}}{=} e^{|\Lambda|\beta h} \mathbf{E}_\Lambda\Big(\prod_{l \geq 1} \Big(2\cosh(l\beta h)\Big)^{\alpha_l(\xi_\beta)}\Big)$$

$$\overset{h=0}{=} \mathbf{E}_\Lambda\Big(2^{\sum_{l \geq 1} \alpha_l(\xi_\beta)}\Big)$$

Notation:

$\alpha_l(\sigma) :=$ number of cylces of length l in $\sigma \in \mathcal{S}_{|\Lambda|}$

$\lambda_x(\sigma) :=$ length of cylce in $\sigma \in \mathcal{S}_\Lambda$ containing $x \in \Lambda$

And a straightforward (but very useful) identity: for $F : \mathbb{N} \to \mathbb{R}$

$$\sum_{l \geq 1} lF(l)\alpha_l(\sigma) = \sum_{x \in \Lambda} F(\lambda_x(\sigma))$$

$$|\Lambda|^{-1}\mathbf{E}_\Lambda\Big(\sum_{l \geq 1} lF(l)\alpha_l(\xi_\beta)\Big) = \mathbf{E}_\Lambda\Big(F(\lambda_0(\sigma))\Big)$$

The spontaneous magnetisation:

$$
\begin{aligned}
m_\Lambda(\beta,2h) &= \frac{1}{\beta|\Lambda|}\frac{\partial \log \Xi_\Lambda}{\partial h}(\beta,2h) - \frac{1}{2} \\
&= \frac{1}{|\Lambda|}\frac{\mathbf{E}_\Lambda\Big(\prod_{l\geq 1}\Big(2\cosh(l\beta h)\Big)^{\alpha_l(\xi_\beta)}\sum_{k\geq 1} k\alpha_k(\xi_\beta)\tanh(k\beta h)\Big)}{\mathbf{E}_\Lambda\Big(\prod_{l\geq 1}\Big(2\cosh(l\beta h)\Big)^{\alpha_l(\xi_\beta)}\Big)} \\
&= \frac{\mathbf{E}_\Lambda\Big(\prod_{l\geq 1}\Big(2\cosh(l\beta h)\Big)^{\alpha_l(\xi_\beta)}\tanh(\lambda_0(\xi_\beta)\beta h)\Big)}{\mathbf{E}_\Lambda\Big(\prod_{l\geq 1}\Big(2\cosh(l\beta h)\Big)^{\alpha_l(\xi_\beta)}\Big)}
\end{aligned}
$$

$$
\begin{aligned}
m(\beta) &= \lim_{h\to 0}\lim_{\Lambda\nearrow\mathbb{Z}^d} m_\Lambda(\beta,h) \\
&= \lim_{n\to\infty}\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{\mathbf{E}_\Lambda\Big(2^{\sum_{l\geq 1}\alpha_l(\xi_\beta)}\mathbb{1}_{\{\lambda_0(\xi_\beta)>n\}}\Big)}{\mathbf{E}_\Lambda\Big(2^{\sum_{l\geq 1}\alpha_l(\xi_\beta)}\Big)}
\end{aligned}
$$

The Long Range Order at $h = 0$:

$$r^2(\beta) := \underbrace{\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \left\langle S_+(0) S_-(x) \right\rangle_{\Lambda,\beta}}_{\text{Heisenberg spins}} \overset{\text{or}}{=} \underbrace{\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \left\langle \mathfrak{b}^\dagger(0) \mathfrak{b}(x) \right\rangle_{\Lambda,\beta}}_{\text{hard core Bose gas}}$$

$$= \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \frac{\mathbf{E}_\Lambda\left(2^{\sum_{l \geq 1} \alpha_l(\xi_\beta)} \mathbb{1}_{\{0,x \text{ on the same cycle of } \xi_\beta\}}\right)}{\mathbf{E}_\Lambda\left(2^{\sum_{l \geq 1} \alpha_l(\xi_\beta)}\right)}$$

$$= \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \frac{\mathbf{E}_\Lambda\left(2^{\sum_{l \geq 1} \alpha_l(\xi_\beta)} \lambda_0(\xi_\beta)\right)}{\mathbf{E}_\Lambda\left(2^{\sum_{l \geq 1} \alpha_l(\xi_\beta)}\right)}$$

The rest is fantasy and conjectures.
The formulas make prefect sense with 2 replaced by $\theta \geq 1$:

$$m_\theta(\beta) = \lim_{n\to\infty} \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\mathbf{E}_\Lambda\Big(\theta^{\sum_{l\geq 1}\alpha_l(\xi_\beta)} \mathbb{1}_{\{\lambda_0(\xi_\beta)>n\}}\Big)}{\mathbf{E}_\Lambda\Big(\theta^{\sum_{l\geq 1}\alpha_l(\xi_\beta)}\Big)}$$

$$r^2_\theta(\beta) := \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \frac{\mathbf{E}_\Lambda\Big(\theta^{\sum_{l\geq 1}\alpha_l(\xi_\beta)} \lambda_0(\xi_\beta)\Big)}{\mathbf{E}_\Lambda\Big(\theta^{\sum_{l\geq 1}\alpha_l(\xi_\beta)}\Big)}$$

though, their interpretation as q-physical objects is less clear.

Compare with the FK, or, random cluster models of
[CM Fortuin, PW Kasteleyn (1972)]: ... (explain) ...
$\theta = 1$ percolation; $\theta = 2$ Ising; $\theta = 3, 4, \dots$ Potts;

$\theta = 1$ is particularly appealing: "quantum percolation"?

Conjectures: Based on [BT (1993)], $\theta = 1, 2$.

- $d = 2$: $\forall \beta < \infty$, $r_\theta(\beta) = 0$, $m_\theta(\beta) = 0$.
- $d \geq 3$: $\exists 0 < \beta_* < \beta^* < \infty$, such that
  - for $\beta < \beta_*$: $r_\theta(\beta) = 0$, $m_\theta(\beta) = 0$.
  - for $\beta > \beta^*$: $r_\theta(\beta) > 0$, $m_\theta(\beta) > 0$.

Some results: Mostly for $\theta = 1$, some extended/able to $\theta = 2$

- [O Schramm (2005)]
  Fully resolved on the $\Lambda_n = K_n$, $n \to \infty$.
- [A Hammond (2015)]
  Essentially resolved on the tree $\mathbb{T}_d$ with $d \gg 1$.
- [R Kotecký, P Miloś, D Ueltschi (2016)]
  Partial results on $\Lambda_n = \{0, 1\}^n$, $n \to \infty$.
- [D Ueltschi (2013-. . .)] Various joint extensions of [BT (1993)] and [M Aizenman, B Nachtergaele (1994)]
- . . . . . .

Nobel Laureates (Physics) who contributed substantially to the subject of the course and appeared in these lectures:

1921: Einstein
1932: Heisenberg
1945: Pauli
1957: Lee, Yang
1965: Feynman
1968: Onsager
1970: Néel
2016: Kosterlitz, Thouless

Could have appeared, left out only due to time constraints:

1952: Bloch
1977: Anderson, Mott
2016: Haldane

In the same league - and appeared in the talk:

Peierls, Dyson