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Statistical Physics with Continuous Symmetries

V. The Quantum Heisenberg Model ctd.
Theorems and Proofs

The Mermin-Wagner Theorem – quantum setting

$$H_{\Lambda,\varepsilon} = -\frac{1}{2} \sum_{x \sim y \in \Lambda} \Big(S_1(x) S_1(y) + S_2(x) S_2(y) + u S_3(x) S_3(y) \Big)$$

$$\underbrace{-h \sum_{x \in \Lambda} S_3(x)}_{\substack{\text{transverse} \\ \text{field term}}} \underbrace{-\varepsilon \sum_{x \in \Lambda} S_1(x)}_{\substack{U(1)\text{-symmetry} \\ \text{breaking term}}}$$

$$= -\frac{1}{2} \sum_{x \sim y \in \Lambda} \Big(S_+(x) S_-(y) + u S_3(x) S_3(y) \Big)$$

$$\underbrace{-h \sum_{x \in \Lambda} S_3(x)}_{\substack{\text{transverse} \\ \text{field term}}} \underbrace{-\frac{\varepsilon}{2} \sum_{x \in \Lambda} (S_+(x) + S_-(x))}_{\substack{U(1)\text{-symmetry} \\ \text{breaking term}}}$$

Theorem 1. [N.D. Mermin, H. Wagner (1966))
In $d = 2$, *at any* $\beta < \infty$

$$\lim_{\varepsilon \to 0} \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle S_{\pm}(x) \rangle_{\Lambda,\varepsilon,\beta} = 0.$$

Proof: We will apply Bogoliubov's inequality with, $p \in \Lambda^*$ fixed:

$$A = \widehat{S}_+(p) = \sum_{x \in \Lambda} e^{ip \cdot x} S_+(x)$$

$$A^* = \widehat{S}_-(-p) = \sum_{x \in \Lambda} e^{-ip \cdot x} S_-(x)$$

$$C = \widehat{S}_3(p) = \sum_{x \in \Lambda} e^{ip \cdot x} S_3(x)$$

$$C^* = \widehat{S}_3(-p) = \sum_{x \in \Lambda} e^{-ip \cdot x} S_3(x)$$

The (anti)commutators involved are:

$$\frac{1}{2}\{A, A^*\} = \widehat{S}_+(p)\widehat{S}_-(-p) - \widehat{S}_3(0)$$

$$[C^*, A] = \widehat{S}_+(0)$$

$$[[C, H_\varepsilon], C^*] = \sum_{x\in\Lambda}\sum_{|e|=1}(1 - \cos p\cdot e)S_+(x)S_-(x+e) + \frac{\varepsilon}{2}\sum_{x\in\Lambda}(S_+(x) + S_-(x))$$

These identities follow from **very instructive computations** left as **(HW)**.

Expectations, correlations: $\beta < \infty$, $h \in \mathbb{R}$ fixed;
$\Lambda \nearrow \mathbb{Z}^d$, then $\varepsilon \to 0$

$$\widehat{c}_{\Lambda,\varepsilon}(p) := |\Lambda|^{-1} \left\langle \widehat{S}_+(p) \widehat{S}_-(-p) \right\rangle_{\Lambda,\varepsilon} \quad \text{correlation}$$

$$= \sum_{x \in \Lambda} e^{ip\cdot x} \left\langle S_+(0) S_-(x) \right\rangle_{\Lambda,\varepsilon}$$

$$\mu_{\Lambda,\varepsilon} := \langle S_3(x) \rangle_{\Lambda,\varepsilon} \quad \text{transversal magnetisation}$$

$$m_{\Lambda,\varepsilon} := \left\langle S_+(x) \right\rangle_{\Lambda,\varepsilon} = \langle S_-(x) \rangle_{\Lambda,\varepsilon} \quad \text{parallel magnetisation}$$

Bogoliubov:

$$\widehat{c}_{\Lambda,\varepsilon}(p) - \mu_{\Lambda,\varepsilon} \geq \frac{m_{\Lambda,\varepsilon}^2}{\beta(\sum_{e:|e|=1} \underbrace{(1-\cos p\cdot e)}_{\geq 0} \underbrace{\left\langle S_+(0)S_-(e)\right\rangle_{\Lambda,\varepsilon}}_{\leq s(s+1)} + \varepsilon \underbrace{m_{\Lambda,\varepsilon}}_{\leq s})}$$

$$\geq \frac{m_{\Lambda,\varepsilon}^2}{\beta\left(D(p)s(s+1)+\varepsilon s\right)}$$

Note: The denominator is a priori positive!

Take on both sides $|\Lambda|^{-1}\sum_{p\in\Lambda^*}\ldots$:

$$\underbrace{\left\langle S_+(0)S_-(0)\right\rangle_{\Lambda,\varepsilon} - \mu_{\Lambda,\varepsilon}}_{\leq s(s+1)+s} \geq \frac{m_{\Lambda,\varepsilon}^2}{\beta}\frac{1}{|\Lambda|}\sum_{p\in\Lambda^*}\frac{1}{s(s+1)D(p)+\varepsilon s}$$

Let $\Lambda \nearrow \mathbb{Z}^d$:

$$s^2 + 2s \geq \frac{\overline{\lim}_{\Lambda \nearrow \mathbb{Z}^d} m^2_{\Lambda,\varepsilon}}{(2\pi)^d \beta} \int_{[-\pi,\pi]^d} \frac{1}{s(s+1)D(p) + \varepsilon s} dp$$

Let $\varepsilon \to 0$:

$$s^2 + 2s \geq \frac{\overline{\lim}_{\varepsilon \to 0} \overline{\lim}_{\Lambda \nearrow \mathbb{Z}^d} m^2_{\Lambda,\varepsilon}}{(2\pi)^d \beta s(s+1)} \underbrace{\int_{[-\pi,\pi]^d} \frac{1}{D(p)} dp}_{=\infty}$$

Hence, necessarily

$$\overline{\lim_{\varepsilon \to 0}} \; \overline{\lim_{\Lambda \nearrow \mathbb{Z}^d}} \, m^2_{\Lambda,\varepsilon} = 0$$

$\square$ M-W

The Dyson-Lieb-Simon Theorem:

$$H_\Lambda = -\frac{1}{2}\sum_{x\sim y\in\Lambda}\Big(S_1(x)S_1(y) + S_2(x)S_2(y) + uS_3(x)S_3(y)\Big)\underbrace{-h\sum_{x\in\Lambda}S_3(x)}_{\substack{\text{transverse}\\ \text{field term}}}$$

$$= -\frac{1}{2}\sum_{x\sim y\in\Lambda}\Big(S_+(x)S_-(y) + uS_3(x)S_3(y)\Big)\underbrace{-h\sum_{x\in\Lambda}S_3(x)}_{\substack{\text{transverse}\\ \text{field term}}}$$

The LRO:

$$r(\beta)^2 := \lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{1}{\Lambda}\sum_{x\in\Lambda}\langle S_1(0)S_1(x)\rangle_{\Lambda,\beta} = \lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{1}{\Lambda}\sum_{x\in\Lambda}\langle S_2(0)S_2(x)\rangle_{\Lambda,\beta}$$

I will mostly concentrate on the isotropic cases with no transverse field, $u = \pm 1$, $h = 0$ and comment on the extensions.

Theorem 2. [F Dyson, EH Lieb, B Simon (1978)], with extensions/additions/improvements from [EJ Neves, JF Perez (1986)], [T Kennedy, EH Lieb, BS Shastry (1988)], [K Kubo, T Kishi (1988)], . . .

(i) [Néel order in the ground state.] $r_\infty > 0$ *if*

$$\circ\, d = 2: \qquad \begin{cases} s = \frac{1}{2}, & h = 0, \quad u \in [-0.13, 0] \\ s \geq 1, & h = 0, \quad u \in [-1, 0] \end{cases}$$

$$\circ\, d \geq 3: \qquad s \geq \frac{1}{2}, \quad h = 0, \quad u \in [-1, 0]$$

(ii) [Néel order at positive temperature.] *There exists* $\beta^* = \beta^*(s, d, u) < \infty$ *such that for* $\beta > \beta_*$, $r_\beta > 0$, *if*

$$\circ\, d \geq 3: \qquad s \geq \frac{1}{2}, \quad h = 0, \quad u \in [-1, 0]$$

Comments & remarks:

(1) Note that in all cases

- $h = 0$: no transverse field
- $u \leq 0$: antiferromagnetic coupling

Extension to $u \in (0, 1]$ (ferromagnetic coupling) and/or $|h| < h_*(d, s, u, \beta)$ (small transverse field) remain *major open problems*.

(2) Historical:

- [DLS (1978)]: $d \geq 3,\ s \geq 1,\ u \in [-1, 0], \quad \beta \in (\beta_*, \infty]$
- [NP (1986)]: $d = 2,\ s \geq 1,\ u \in [-1, 0], \quad \beta = \infty$
- [KLS (1988)]: $d \geq 3,\ s = \frac{1}{2},\ u \in [-1, 0], \quad \beta \in (\beta_*, \infty]$
- [KK (1988)]: $d = 2,\ s = \frac{1}{2},\ u \in [-0.13, 0],\ \beta = \infty$

(3) $|u| > 1$ is different: at low temperature *Ising-like* phase transition expected, proved for $u > 1$ [T Kennedy (1985)]

A warning sign: The classical IRB reasoning must fail, due to quantum ground state fluctuations! Here is an instructive example showing this.

Let $s = \frac{1}{2}$, $u = 1$ (isotropic QHF), $h = 0$ (no transverse field).

$$c_{\Lambda,\beta}(x) := \left\langle \vec{S}(0) \cdot \vec{S}(x) \right\rangle_{\Lambda,\beta} \qquad \widehat{c}_{\Lambda,\beta}(p) := \sum_{x \in \Lambda} e^{ip\cdot x} c_{\Lambda,\beta}(x)$$

and assume (for the sake of the argument) the IRB

$$\widehat{c}_{\Lambda,\beta}(p) \leq \frac{\mathsf{K}}{\beta D(p)}, \qquad \text{with some } K < \infty.$$

Then

$$\left|c_{\Lambda,\beta}(0) - c_{\Lambda,\beta}(x)\right| = |\Lambda|^{-1} \left| \sum_{p\in\Lambda^*} \left(1 - e^{ip\cdot x}\right) \widehat{c}_{\Lambda,\beta}(p) \right|$$

$$= |\Lambda|^{-1} \left| \sum_{p\in\Lambda^*} (1 - \cos(p\cdot x))\, \widehat{c}_{\Lambda,\beta}(p) \right|$$

$$\leq \frac{K}{\beta} |\Lambda|^{-1} \sum_{p\in\Lambda^*} \frac{(1 - \cos(p\cdot x))}{D(p)}$$

$$\overset{\Lambda \nearrow \mathbb{Z}^d}{\longrightarrow} \frac{K}{(2\pi)^d \beta} \underbrace{\int_{[-\pi,\pi]^d} \frac{(1 - \cos(p\cdot x))}{D(p)} dp}_{<\infty \text{ in all } d}$$

$$\overset{\beta\to\infty}{\longrightarrow} 0.$$

However, the following *operator identities/inequalities* hold:

$$\vec{S}(0)\cdot\vec{S}(0) = S_1(0)^2 + S_2(0)^2 + S_3(0)^2 = \frac{3}{4}I$$

$$\vec{S}(0)\cdot\vec{S}(x) = \underbrace{S_1(0)S_1(x) + S_2(0)S_2(x) + S_3(0)S_3(x)}_{I\otimes\cdots\otimes I\otimes(S_1\otimes S_1+S_2\otimes S_2+S_3\otimes S_3)\otimes I\otimes\cdots\otimes I} \le \frac{1}{4}I \quad \text{(HW)}$$

and hence, for all $\beta \geq 0$, and Λ,

$$c_{\Lambda,\beta}(0) - c_{\Lambda,\beta}(x) = \left\langle \vec{S}(0)\cdot\vec{S}(0) - \vec{S}(0)\cdot\vec{S}(x)\right\rangle_{\Lambda,\beta} \geq \frac{1}{2}$$

Modified - more sophisticated - strategy of proof is needed. IRB will be proved *for the Duhamel two-point function* rather than for the correlation function.

I will focus on the isotropic QHAF/QHF, $u = \mp 1$, with no transverse field, $h = 0$.

$$H_\Lambda = -\frac{1}{2} \sum_{x \sim y \in \Lambda} \Big(S_1(x) S_1(y) + S_2(x) S_2(y) \mp S_3(x) S_3(y) \Big)$$

Results are *proved only for the antiferromagnetic coupling*, $u = -1$. However, I want to point out what is the issue with the ferromagnetic case $u = +1$.

In all forthcoming formulas the upper branch stands for the AFM (IRB proved), the lower branch stands for the FM (IRB not proved, but some version expected to hold)

Correlation functions:

Conventional:

$$c_{\Lambda,\beta}(x) := \langle S_1(0)S_1(x)\rangle_{\Lambda,\beta} = \langle S_2(0)S_2(x)\rangle_{\Lambda,\beta}$$

$$\underbrace{=}_{u=\mp 1} (\mp 1)^{|x|} \langle S_3(0)S_3(x)\rangle_{\Lambda,\beta}$$

Duhamel:

$$d_{\Lambda,\beta}(x) := (S_1(0), S_1(x))_{\Lambda,\beta} = (S_2(0), S_2(x))_{\Lambda,\beta}$$

$$\underbrace{=}_{u=\mp 1} (\mp 1)^{|x|} (S_3(0), S_3(x))_{\Lambda,\beta}$$

Bogoliubov:

$$b_{\Lambda,\beta}(x) := \langle [S_1(0), [H_\Lambda, S_1(x)]]\rangle_{\Lambda,\beta} = \langle [S_2(0), [H_\Lambda, S_2(x)]]\rangle_{\Lambda,\beta}$$

$$\underbrace{=}_{u=\mp 1} (\mp 1)^{|x|} \langle [S_3(0), [H_\Lambda, S_3(x)]]\rangle_{\Lambda,\beta}$$

And their Fourier transforms:

$$\widehat{c}_{\Lambda,\beta}(p) := \sum_{x\in\Lambda} e^{ip\cdot x} c_{\Lambda,\beta}(x) = \frac{1}{|\Lambda|}\left\langle \widehat{S}_1(-p)\widehat{S}_1(p)\right\rangle_{\Lambda,\beta}$$

$$\widehat{d}_{\Lambda,\beta}(p) := \sum_{x\in\Lambda} e^{ip\cdot x} d_{\Lambda,\beta}(x) = \frac{1}{|\Lambda|}\left(\widehat{S}_1(p), \widehat{S}_1(p)\right)_{\Lambda,\beta}$$

$$\widehat{b}_{\Lambda,\beta}(p) := \sum_{x\in\Lambda} e^{ip\cdot x} b_{\Lambda,\beta}(x) = \frac{1}{|\Lambda|}\left\langle \left[\widehat{S}_1(-p), \left[H_\Lambda, \widehat{S}_1(p)\right]\right]\right\rangle_{\Lambda,\beta}$$

Note. These are correlation type objects. In particular, for all $p \in \Lambda^*$

$$\widehat{c}_{\Lambda,\beta}(p) \geq 0, \qquad \widehat{d}_{\Lambda,\beta}(p) \geq 0, \qquad \widehat{b}_{\Lambda,\beta}(p) \geq 0 \qquad \text{(HW)}$$

Proof of DLS Theorem:

Step 1. **Apply Falk-Bruch inequality** with

$$A = \widehat{S}_1(p), \qquad A^* = \widehat{S}_1(-p)$$

Get

$$\widehat{c}_{\Lambda,\beta}(p) \leq \frac{\beta \widehat{b}_{\Lambda,\beta}(p)}{4} \phi \left(\frac{4\widehat{d}_{\Lambda,\beta}(p)}{\beta \widehat{b}_{\Lambda,\beta}(p)} \right) \tag{1}$$

Step 2. **Compute the double commutators.** (HW):
These computations are simpler in the *isotropic* cases $u = \mp 1$.

$$[S_1(0), [H_\Lambda, S_1(x)]] =$$

$$= \begin{cases} \sum_{|e|=1} \left(S_2(0)S_2(e) \mp S_3(0)S_3(e) \right) & \text{if } |x-0| = 0 \\ -\left(S_3(0)S_3(x) \mp S_2(0)S_2(x) \right) & \text{if } |x-0| = 1 \\ 0 & \text{if } |x-0| > 1 \end{cases}$$

Assume $\Lambda = (\mathbb{Z}/L)^d$ cubic. This is not essential, but simplifies the computations. Denote

$$\kappa_{\Lambda,\beta} := \langle S_1(0)S_1(e) \rangle_{\Lambda,\beta} = \langle S_2(0)S_2(e) \rangle_{\Lambda,\beta} = \mp \langle S_3(0)S_3(e) \rangle_{\Lambda,\beta}$$

The r.h.s. is the same for all $e \in \mathbb{Z}^d$, $|e| = 1$.

Get

$$b_{\Lambda,\beta}(x) = \begin{cases} 4d\kappa_{\Lambda,\beta} & \text{if } |x-0| = 0 \\ \pm 2\kappa_{\Lambda,\beta} & \text{if } |x-0| = 1 \\ 0 & \text{if } |x-0| > 1 \end{cases}$$

and, finally,

$$\widehat{b}_{\Lambda,\beta}(p) = \begin{cases} 4\kappa_{\Lambda,\beta} \underbrace{(2d - D(p))}_{\widetilde{D}(p)} & \text{for the QHAF} \\ 4\kappa_{\Lambda,\beta} D(p) & \text{for the QHF} \end{cases} \tag{2}$$

It also follows that

$$\kappa_{\Lambda,\beta} \geq 0.$$

See sketchy plots of the functions $p \mapsto D(p), \widetilde{D}(p)$ on next slides.

FOOPLOT

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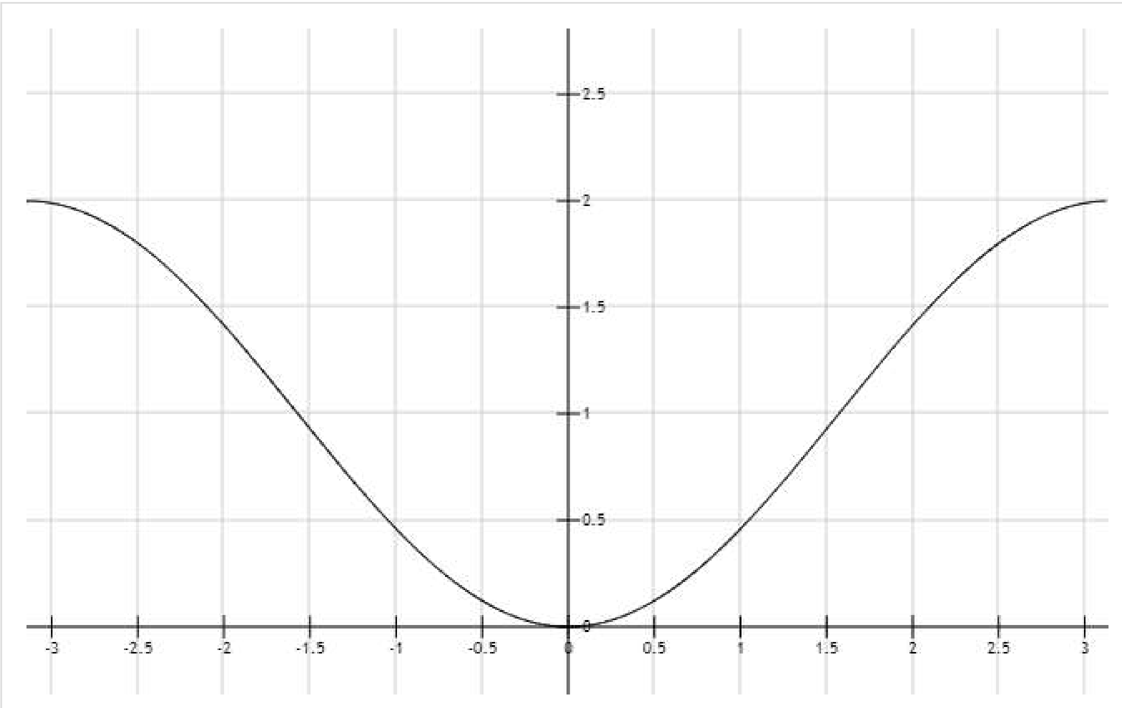

Function y(x)

1-cos(x)

Function

Add

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Grid Spacing

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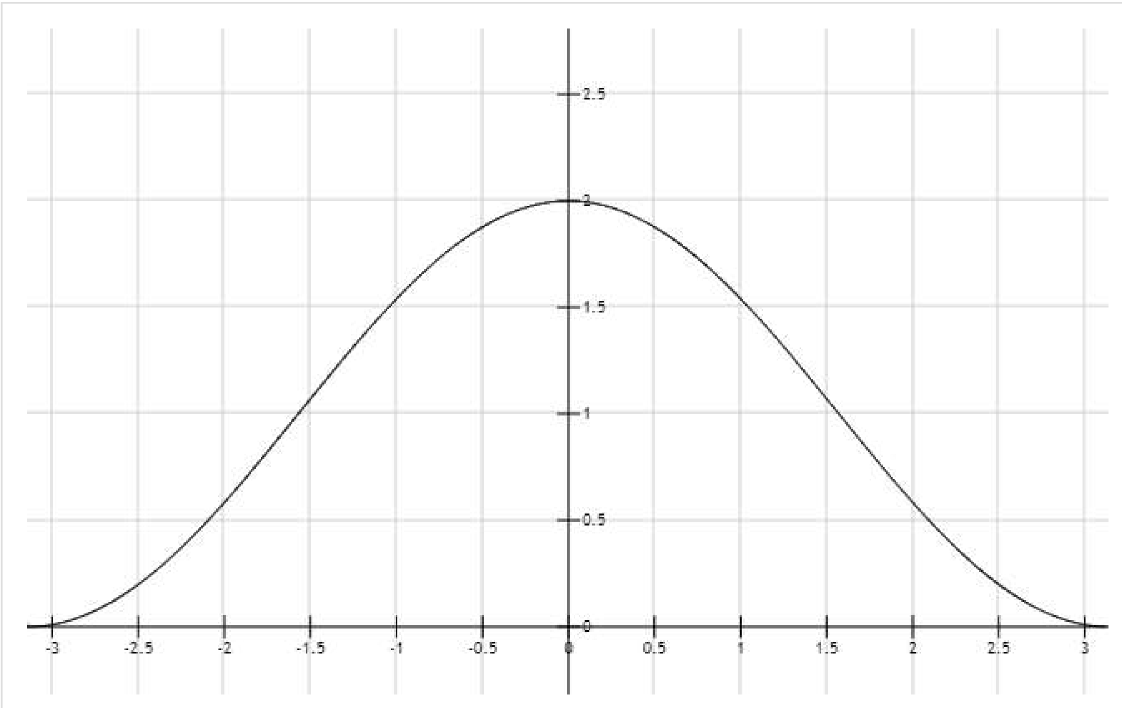


Function y(x)

1+cos(x)

Function

Add

Tweet

Window

Display

Grid Spacing

Step 3. **IRB for the Duhamel two point function.**

$$\widehat{d}_{\Lambda,\beta}(p) \leq \frac{1}{2\beta D(p)} \tag{3}$$

Proved only for AF coupling, $u \leq 0$, and no transverse field, $h = 0$. This is an essential restriction of the proof method.

For ferromagnetic coupling, $u \in (0, 1]$, and small transversal field $|h| < h_*(d, s, \beta)$ something of the form

$$\widehat{d}_{\Lambda,\beta}(p) \leq \frac{K}{2\beta D(p)} \tag{4}$$

may be expected, where $K = K(d, s, h)$. This is THE major open problem in this context.

Note the difference from the classical/commutative setting: The IRB holds for the Duhamel correlation $\widehat{d}_{\Lambda,\beta}$ rather than the conventional one.

Proof of (3) postponed to the end.

Step 4. **Put these things together.**

$$\widehat{c}_{\Lambda,\beta}(p) \underbrace{\leq}_{\text{F-B}} \frac{\beta\widehat{b}_{\Lambda,\beta}(p)}{4}\phi\left(\frac{4\widehat{d}_{\Lambda,\beta}(p)}{\beta\widehat{b}_{\Lambda,\beta}(p)}\right)$$

$$\underbrace{\leq}_{\text{IRB}} \frac{\beta\widehat{b}_{\Lambda,\beta}(p)}{4}\phi\left(\frac{2}{\beta^2 D(p)\widehat{b}_{\Lambda,\beta}(p)}\right)$$

$$\underbrace{=}_{\text{comp.}} \begin{cases} \beta\kappa_{\Lambda,\beta}\widetilde{D}(p)\phi\left(\frac{1}{2\beta^2\kappa_{\Lambda,\beta}D(p)\widetilde{D}(p)}\right) & \text{QHAF, proved} \\ \beta\kappa_{\Lambda,\beta}D(p)\phi\left(\frac{1}{2\beta^2\kappa_{\Lambda,\beta}D(p)^2}\right) & \text{QHF, presumed} \end{cases} \tag{5}$$

Recall:

$$\kappa_{\Lambda,\beta} \in [0, s^2]$$

Step 5. **Two identities ("sum rules").**

$$\frac{s(s+1)}{3} = c_{\Lambda,\beta}(0) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \widehat{c}_{\Lambda,\beta}(p)$$

$$= r_{\Lambda,\beta}^2 + \frac{1}{|\Lambda|} \sum_{p \in \Lambda^* \setminus \{0\}} \widehat{c}_{\Lambda,\beta}(p) \tag{6}$$

$$\kappa_{\Lambda,\beta} = \frac{1}{2d} \sum_{|e|=1} c_{\Lambda,\beta}(e) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \left(1 - \frac{D(p)}{d}\right) \widehat{c}_{\Lambda,\beta}(p)$$

$$= r_{\Lambda,\beta}^2 + \frac{1}{|\Lambda|} \sum_{p \in \Lambda^* \setminus \{0\}} \left(1 - \frac{D(p)}{d}\right) \widehat{c}_{\Lambda,\beta}(p)$$

$$\leq r_{\Lambda,\beta}^2 + \frac{1}{|\Lambda|} \sum_{p \in \Lambda^* \setminus \{0\}} \left(1 - \frac{D(p)}{d}\right)_+ \widehat{c}_{\Lambda,\beta}(p) \tag{7}$$

Step 6. **Long Range Order in the Ground State.**

First $\beta \to \infty$ (finite volume ground state), no issues . . .
then $\Lambda \nearrow \mathbb{Z}^d$ (thermodynamic limit), subsequential limits

Remark 1. The ground states of the isotropic ferromagnetic QHM, $u = +1$, $h = 0$, on any graph, are well understood (somewhat trivial): They are all vectors in the $(2|\Lambda| s + 1)$-dimensional subspace of maximal total spin. (HW)

$$\{\varphi \in \otimes_{x\in\Lambda}\mathbb{C}^{2s+1} : (\mathbb{S}_1^2 + \mathbb{S}_2^2 + \mathbb{S}_3^2)\varphi = (|\Lambda| s)(|\Lambda| s + 1)\varphi\}.$$

The ground state LRO is

$$r^2_{\Lambda,\infty} := \lim_{\beta\to\infty} r^2_{\Lambda,\beta} = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\lim_{\beta\to\infty}\langle S_1(0)S_1(x)\rangle_{\Lambda,\beta} = \frac{s^2}{3} - \frac{s}{3|\Lambda|} \text{ (HW)}$$

$$r^2_\infty := \lim_{\Lambda\nearrow\mathbb{Z}^d} r^2_{\Lambda,\infty} = \frac{s^2}{3}$$

Remark 2. The ground state(s) of the isotropic QHAF, $u = -1$, $h = 0$ is(are) very complicated. [W Marshall (1955)], [EH Lieb, DC Mattis (1962)]: on a bipartite lattice it is unique, nondegenerate. (Perron-Frobenius argument.)

Proceed with the QHAF. Letting $\beta \to \infty$ in the upper branch of (5) and in (6) and (7) get:

$$\widehat{c}_{\Lambda,\infty}(p) \leq \sqrt{\frac{\kappa_{\Lambda,\infty}}{2}}\sqrt{\frac{\widetilde{D}(p)}{D(p)}}$$

$$r^2_{\Lambda,\infty} \geq \frac{s(s+1)}{2} - \sqrt{\frac{\kappa_{\Lambda,\infty}}{2}}\frac{1}{|\Lambda|}\sum_{p\in\Lambda^*\setminus\{0\}}\sqrt{\frac{\widetilde{D}(p)}{D(p)}}$$

$$r^2_{\Lambda,\infty} \geq 2\sqrt{\frac{\kappa_{\Lambda,\infty}}{2}}\left(\sqrt{\frac{\kappa_{\Lambda,\infty}}{2}} - \frac{1}{2}\frac{1}{|\Lambda|}\sum_{p\in\Lambda^*\setminus\{0\}}\left(1 - \frac{D(p)}{d}\right)_+\sqrt{\frac{\widetilde{D}(p)}{D(p)}}\right)$$

Now, take the thermodynamic (subsequential) limit $\Lambda \nearrow \mathbb{Z}^d$ and get

$$r_\infty^2 \geq \frac{s(s+1)}{3} - \sqrt{\frac{\kappa_\infty}{2}} J, \qquad r_\infty^2 \geq 2\sqrt{\frac{\kappa_\infty}{2}} \left(\sqrt{\frac{\kappa_\infty}{2}} - \frac{1}{2} I \right), \tag{8}$$

where

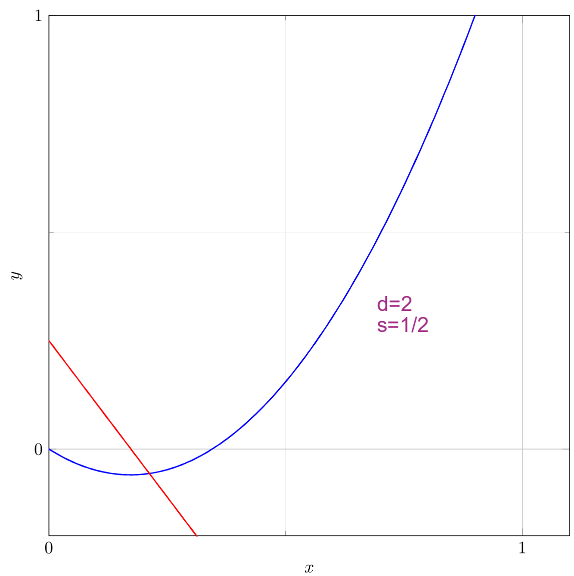
$$J = J(d) := \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \sqrt{\frac{\widetilde{D}(p)}{D(p)}} dp$$

$$I = I(d) := \int_{[-\pi,\pi]^d} \left(1 - \frac{D(p)}{d} \right)_+ \sqrt{\frac{\widetilde{D}(p)}{D(p)}} dp$$

Finally, the ineqs (8) jointly imply that $r_\infty > 0$ is guaranteed if

$$\frac{3}{2} I(d) J(d) < s(s+1)$$

See the plots on next page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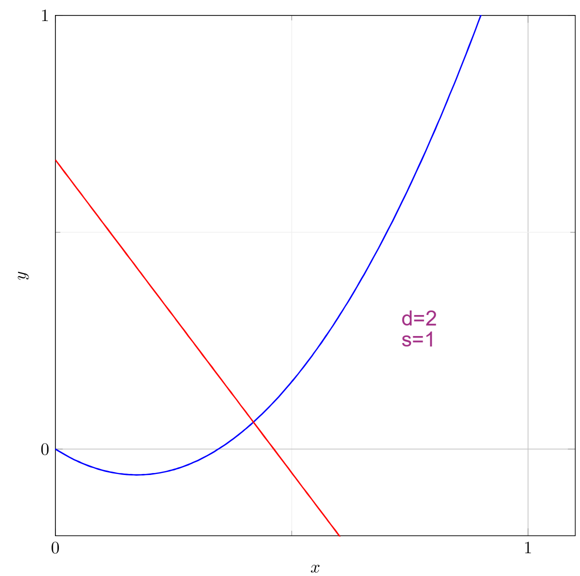
0
y
d=2
s=1/2
0
1
x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y
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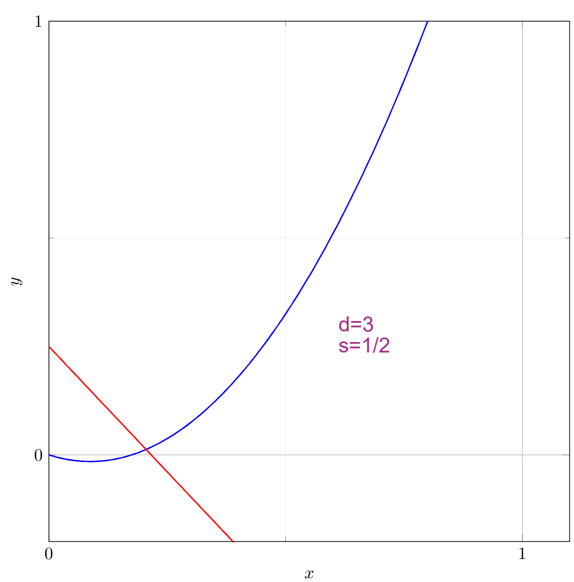
1
0
y
d=3
s=1/2
0
1
x

	$I(d)$	$J(d)$	$\frac{3}{2}I(d)J(d)$
$d = 2$	0.690	1.443	$1.494 \in (\frac{3}{4}, 2)$
$d = 3$	0.350	1.157	$0.607 < \frac{3}{4}$

Conclusion:

- $d = 2$: The isotropic QHAF exhibits Néel order in the ground state for $s = 1, \frac{3}{2}, 2, \ldots$. The case of $s = \frac{1}{2}$ remains open.
- $d = 3$: The isotropic QHAF exhibits Néel order in the ground state for $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$.
- In the non-isotropic cases, $u \in (-1, 0]$, the computational parts are more subtle. For the results see the comments page 10. of these notes.

Step 6. **Stability of the ground state LRO in** $d \geq 3$

This will be essentially a dominated convergence argument.

Apply the upper bound (5) in the identities (6) and (7)

$$r^2_{\Lambda,\beta} \geq \frac{s(s+1)}{3} - \frac{1}{|\Lambda|} \sum_{p \in \Lambda^* \setminus \{0\}} \beta \kappa_{\Lambda,\beta} \widetilde{D}(p) \phi \left(\frac{1}{2\beta^2 \kappa_{\Lambda,\beta} D(p) \widetilde{D}(p)} \right)$$

$$r^2_{\Lambda,\beta} \geq \kappa_{\Lambda,\beta} - \frac{1}{|\Lambda|} \sum_{p \in \Lambda^* \setminus \{0\}} \left(1 - \frac{D(p)}{d} \right)_+ \beta \kappa_{\Lambda,\beta} \widetilde{D}(p) \phi \left(\frac{1}{2\beta^2 \kappa_{\Lambda,\beta} D(p) \widetilde{D}(p)} \right)$$

and then take the thermodynamic (subseq.) limit $\Lambda \nearrow \mathbb{Z}^d$ to get

$$r^2_{\beta} \geq \frac{s(s+1)}{3} - \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \beta \kappa_{\beta} \widetilde{D}(p) \phi \left(\frac{1}{2\beta^2 \kappa_{\beta} D(p) \widetilde{D}(p)} \right) dp$$

$$r^2_{\beta} \geq \kappa_{\beta} - \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \left(1 - \frac{D(p)}{d} \right)_+ \beta \kappa_{\beta} \widetilde{D}(p) \phi \left(\frac{1}{2\beta^2 \kappa_{\beta} D(p) \widetilde{D}(p)} \right) dp$$

Letting $\beta \to \infty$ *under the integrals* we obtain exactly the inequalities (8) for which we know the consequences. The question remains whether on the r.h.s. the limit $\beta \to \infty$ can be properly done outside the integrals. This is a Dominated Convergence question.

Note that $\phi(y) \leq a\sqrt{y} + by$ for some $a, b \in (0, \infty)$ and hence, the integrands on the r.h.s. are dominated by

$$a' \sqrt{\frac{\widetilde{D}(p)}{D(p)}} + b' \frac{1}{D(p)}$$

uniformly in $\beta > 1$, which is *integrable in* $d \geq 3$ (but not in $d = 2$).

Conclusion: The Néel order of the ground state is stable under small thermal fluctuations ($\beta > \beta^*$) in dimensions $d \geq 3$.

$\square$Thm DLS, modulo IRB

Back to *Step 3:* Proof of the **IRB**

$$\widehat{d}_{\Lambda,\beta}(p) \le \frac{1}{2\beta D(p)} \tag{3}$$

We will proceed similarly as in the classical case (in the proof of Thm FSS). However, there are some substantial differences due to non-commutativity.

$$H_\Lambda := -\frac{1}{2}\sum_{x\sim y\in\Lambda}\Big(S_1(x)S_1(y)+S_2(x)S_2(y)+uS_3(x)S_3(y)\Big)-\sum_{x\in\Lambda}hS_3(x)$$

$$= \;\; \frac{1}{4}\sum_{x\sim y\in\Lambda}\Big((S_1(x)-S_1(y))^2+(S_2(x)-S_2(y))^2+u(S_3(x)-S_3(y))^2\Big)$$

$$-\sum_{x\in\Lambda}\Big(hS_3(x)+d(S_1(x)^2+S_2(x)^2+S_3(x)^2))$$

for $v : \Lambda \to \mathbb{R}$ let

$$H_\Lambda(\underline{v}) := \frac{1}{4} \sum_{x \sim y \in \Lambda} \Big(((S_1(x) + v(x)) - (S_1(y) + v(y))^2 + (S_2(x) - S_2(y))^2 + u(S_3(x) - S_3(y))^2 \Big) - \sum_{x \in \Lambda} \Big(hS_3(x) + d(S_1(x)^2 + S_2(x)^2 + uS_3(x)^2)\Big)$$

$$Z_\Lambda(\underline{v}) := \mathrm{Tr}\,(\exp(-\beta H_\Lambda(\underline{v})))$$

Note that $Z_\Lambda(\underline{0}) = Z_\Lambda$ is the partition function.

Straightforward computation yields:

$$\frac{1}{Z_\Lambda} \frac{\partial^2 Z_\Lambda}{\partial v(x) \partial v(y)}\bigg|_{\underline{v}=\underline{0}} = \sum_{z_1, z_2 \in \Lambda} \Delta_{x,z_1} d(z_1 - z_2) \Delta_{z_2,y} + \frac{1}{\beta} \Delta_{x,y}$$

Note the difference from the classical/commutative setting: on the right hand side of the identity the *Duhamel two-point function* $d(z_1 - z_2)$ replaces the conventional correlation function $c(z_1 - z_2)$!

Theorem 3. [GD & IRB – Quantum Setting] *For any* even Λ, $\beta < \infty$, $\underline{v} \in (\mathbb{R}^\nu)^\Lambda$, $u \leq 0$ *and* $h = 0$ *the following are true*

(i) [Gaussian Domination - 1]

$$Z_{\Lambda,\beta}(\underline{v}) \leq Z_{\Lambda,\beta}(\underline{0})$$

(ii) [Gaussian Domination - 2]

$$\left\langle e^{\underline{v}\cdot\Delta\underline{\sigma}} \right\rangle_{\Lambda,\beta} \leq e^{-\frac{1}{2\beta}\underline{v}\cdot\Delta\underline{v}} \qquad (\mathbf{GD})$$

(iii) [Infrared Bound - 1]*: The* $\Lambda \times \Lambda$ *matrix*

$$\left(\frac{-\partial^2 Z_\Lambda}{\partial v(x) \partial v(y)} \bigg|_{\underline{v}=\underline{0}} \right)_{x,y \in \Lambda}$$

is positive semidefinite.

(iv) [Infrared Bound - 2]*: For all* $p \in \Lambda^* \setminus \{0\}$

$$\hat{d}(p) \leq \frac{1}{2\beta D(p)} \qquad \textbf{(IRB)}$$

Proof of Theorem 3 [Gaussian Domination]:
As in the classical/commutative setting, the following equivalences/implications hold

$$(i) \underbrace{\Leftrightarrow}_{\text{strfwd}} (ii) \underbrace{\Rightarrow}_{\text{expansion}} (iii) \underbrace{\Leftrightarrow}_{\text{FT}} (iv)$$

We will prove *(i)*.

Proposition. [Reflection Positivity - Quantum Setting]
Let $I, A, B, C_1, \ldots, C_l, D_1, \ldots, D_l$ *ne* $m \times m$ *complex matrices. The following inequality holds>*

$$\left| \mathrm{Tr}\exp\left(A \otimes I + I \otimes B - \frac{1}{2}\sum_{k=1}^{l} (C_k \otimes I - I \otimes D_k)^2 \right) \right|^2 \leq \qquad (\mathbf{RP})$$

$$\mathrm{Tr}\exp\left(A \otimes I + I \otimes \overline{A} - \frac{1}{2}\sum_{k=1}^{l} \left(C_k \otimes I - I \otimes \overline{C}_k\right)^2 \right)$$

$$\times\, \mathrm{Tr}\exp\left(\overline{B} \otimes I + I \otimes B - \frac{1}{2}\sum_{k=1}^{l} \left(\overline{D}_k \otimes I - I \otimes D_k\right)^2 \right)$$

Note: On the rhs the complex conjugate matrices appear. Not the adjoints!

Proof of (RP/Q) I will prove it for $l = 1$. ...

$$\left|\mathrm{Tr} e^{A\otimes I + I\otimes B - \frac{1}{2}(C\otimes I - I\otimes D)^2}\right|^2$$

$$\stackrel{1}{=} \lim_{n\to\infty}\left|\mathrm{Tr}\left(e^{\frac{1}{n}A\otimes I}\ e^{\frac{1}{n}I\otimes B}\ e^{-\frac{1}{2n}(C\otimes I - I\otimes D)^2}\right)^n\right|^2$$

$$\stackrel{2}{=} \lim_{n\to\infty}\left|\int_{-\infty}^{\infty} d\Phi(\xi_1)\ldots\int_{-\infty}^{\infty} d\Phi(\xi_n)\ \mathrm{Tr}\overrightarrow{\prod_{r=1}^{n}}\left(e^{\frac{1}{n}A\otimes I}\ e^{\frac{1}{n}I\otimes B}\ e^{\frac{i\xi_r}{\sqrt{n}}(C\otimes I - I\otimes D)}\right)\right|^2$$

$$\stackrel{3}{=} \lim_{n\to\infty}\left|\int_{-\infty}^{\infty} d\Phi(\xi_1)\ldots\int_{-\infty}^{\infty} d\Phi(\xi_n)\ \mathrm{tr}\overrightarrow{\prod_{r=1}^{n}}\left(e^{\frac{1}{n}A}\ e^{\frac{i\xi_r}{\sqrt{n}}C}\right)\mathrm{tr}\overrightarrow{\prod_{r=1}^{n}}\left(e^{\frac{1}{n}B}\ e^{\frac{-i\xi_r}{\sqrt{n}}D}\right)\right|^2$$

$$\stackrel{4}{\leq} \lim_{n\to\infty}\int_{-\infty}^{\infty} d\Phi(\xi_1)\ldots\int_{-\infty}^{\infty} d\Phi(\xi_n)\ \mathrm{tr}\overrightarrow{\prod_{r=1}^{n}}\left(e^{\frac{1}{n}A}\ e^{\frac{i\xi_r}{\sqrt{n}}C}\right)\mathrm{tr}\overrightarrow{\prod_{r=1}^{n}}\left(e^{\frac{1}{n}\overline{A}}\ e^{\frac{-i\xi_r}{\sqrt{n}}\overline{C}}\right)\times$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} d\Phi(\xi_1)\ldots\int_{-\infty}^{\infty} d\Phi(\xi_n)\ \mathrm{tr}\overrightarrow{\prod_{r=1}^{n}}\left(e^{\frac{1}{n}\overline{B}}\ e^{\frac{i\xi_r}{\sqrt{n}}\overline{D}}\right)\mathrm{tr}\overrightarrow{\prod_{r=1}^{n}}\left(e^{\frac{1}{n}B}\ e^{\frac{-i\xi_r}{\sqrt{n}}D}\right)$$

$$\stackrel{\widetilde{3}}{=}\ldots\stackrel{\widetilde{2}}{=}\ldots\stackrel{\widetilde{1}}{=}\ldots$$

Notation on the previous page:

$$d\Phi(\xi) := \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi, \qquad \overrightarrow{\prod_{r=1}^{n}} X_r := X_1 X_2 \cdots X_r$$

(1) & ($\tilde{1}$): Use Lie product

$$e^{A+B} = \lim_{n\to\infty} \left(e^{A/n} e^{B/n}\right)^n$$

(2) & ($\tilde{2}$): Use Gaussian integrals

$$e^{-M^2/2} = \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} e^{i\xi M}$$

(3) & ($\tilde{3}$): Use $(A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I)$ and

$$\mathrm{Tr}(A \otimes B) = \mathrm{tr}A \ \mathrm{tr}B,$$

(3) Use Schwarz. $\square$(RP)

Back to the proof of GD/Q:
We proceed very similarly as in the classical/commutative setting. However, there will be one surprise.

Assume that the discrete torus Λ is of even side-lengths. Divide Λ in two symmetric halves by a hyperplane intersecting (cutting) only edges

$$\Lambda = \Lambda_{\text{right}} \cup \Lambda_{\text{left}}$$

and define the natural reflection through the dividing hyperplane

$$R : \Lambda \to \Lambda$$

We apply the RP/Q Lemma in the following setting

$$\mathcal{H} = \mathcal{H}_{\text{right}} \otimes \mathcal{H}_{\text{left}} = \left(\mathbb{C}^{2s+1}\right)^{\otimes \Lambda_{\text{right}}} \otimes \left(\mathbb{C}^{2s+1}\right)^{\otimes \Lambda_{\text{left}}}$$

$$A = -\frac{\beta}{4} \sum_{x \sim y \in \Lambda_{\text{right}}} \Big(((S_1(x) + v(x)) - (S_1(y) + v(y))^2 + (S_2(x) - S_2(y))^2 + u(S_3(x) - S_3(y))^2\Big)$$

$$+ \beta \sum_{x \in \Lambda_{\text{right}}} \Big(hS_3(x) + d(S_1(x)^2 + S_2(x)^2 + uS_3(x)^2))$$

$B =$ same with Λ_{left}

$$k = \big(x \sim y, \alpha\big) : \quad x \in \Lambda_{\text{right}}, \; y \in \Lambda_{\text{left}}, \; \alpha = 1, 2, 3$$

$$C_k = \begin{cases} \sqrt{\frac{\beta}{2}}\,(S_1(x) + v(x)) \\ \sqrt{\frac{\beta}{2}} S_2(x) \\ \sqrt{\frac{\beta}{2}}\sqrt{u} S_3(x) \end{cases} \qquad D_k = \begin{cases} \sqrt{\frac{\beta}{2}}\,(S_1(y) + v(y)) \\ \sqrt{\frac{\beta}{2}} S_2(y) \\ \sqrt{\frac{\beta}{2}}\sqrt{u} S_3(y) \end{cases}$$

The RP/Q Lemma can be applied if and only if all matrices on the previous page have **jointly real representation**. However, (S_1, S_2, S_3) have only $(\mathbb{R}, \mathbb{R}, i\mathbb{R})$ (or mixed) representation. For this reason $h = 0$ and $u \leq 0$ must be imposed in order that RP/Q could be applied.

The rest of the proof is identical to that in the classical.commutative setting (see the proof of Thm FSS).

$\square$GD/Q