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## **Statistical Physics with Continuous Symmetries**

### **III. The Quantum Heisenberg Model**

# The Quantum vs. Classical Setting

	Classical	Quantum
State space	$(\Omega, \mathcal{F}, \mu)$	$\mathcal{H}$
Observables	$A : \Omega \rightarrow \mathbb{R}$	$A = A^* \in \mathcal{B}(\mathcal{H})$
Hamiltonian	$H : \Omega \rightarrow \mathbb{R}$	$H = H^* \in \mathcal{B}(\mathcal{H})$
Symmetries	$\mathcal{G} \ni g \mapsto U_g : \Omega \rightarrow \Omega$ $\mu(U_g \cdot) = \mu(\cdot)$ $U_{g \cdot g'} = U_g \circ U_{g'}$ , $H(U_g \omega) = H(\omega)$	$\mathcal{G} \ni g \mapsto U_g \in \mathcal{U}(\mathcal{H})$ $U_{g^{-1}} = U_g^*$ $U_{g \cdot g'} = U_g U_{g'}$ $U_g H U_g^* = H$

	Classical	Quantum
Gibbs weights	$e^{-\beta H}$	$e^{-\beta H}$
Partition fnc	$Z_\beta := \int_{\Omega} e^{-\beta H(\omega)} \mu(d\omega)$	$Z_\beta := \text{Tr}(e^{-\beta H})$
Thermal ave	$\underbrace{\frac{1}{Z_\beta} \int_{\Omega} e^{-\beta H(\omega)} A(\omega) \mu(d\omega)}_{\langle A \rangle_\beta}$	$\underbrace{\frac{1}{Z_\beta} \text{Tr}(e^{-\beta H} A)}_{\langle A \rangle_\beta}$

A warning sign: Moment generating fnc may cause surprises!

$$\frac{1}{Z_\beta} \frac{\partial^2}{\partial u \partial v} \int_{\Omega} e^{-\beta H + uA + vB} d\mu \Big|_{u=v=0} = \langle AB \rangle_\beta \quad (\text{cl})$$

$$\frac{1}{Z_\beta} \frac{\partial^2}{\partial u \partial v} \text{Tr}(e^{-\beta H + uA + vB}) \Big|_{u=v=0} =: (A, B)_\beta \neq \langle AB \rangle_\beta \quad (\text{qu})$$

## The Lie groups $SO(3)$ , $SU(2)$ and Lie algebra $su(2)$

$$SO(3) = \{A \in \mathbb{R}^{3 \times 3} : AA^\dagger = I, \det A = 1\}$$

$$SU(2) = \{A \in \mathbb{C}^{2 \times 2} : AA^* = I, \det A = 1\}$$

$$so(3) = \{A \in \mathbb{R}^{3 \times 3} : A + A^\dagger = 0\}$$

$$su(2) = \{A \in \mathbb{C}^{2 \times 2} : A - A^* = 0, \text{tr} A = 0\}$$

- $SU(2)$  is the *universal covering group* of  $SO(3)$
- $su(2)$  is the *tangent Lie algebra* of  $SU(2)$ : " $SU(2) = e^{isu(2)}$ "

## Spin representations of $su(2)$ and Pauli matrices

For  $s \in \mathbb{N}/2$  fixed, there exists a unique (up to unitary equivalence) irreducible representation of  $su(2)$  over  $\mathbb{C}^{2s+1}$ :

It's three generators:  $S_1, S_2, S_3$  are *uniquely* (up to unitary equiv.) determined by the relations

$$[S_\alpha, S_\beta] = i\epsilon_{\alpha,\beta,\gamma}S_\gamma, \quad S_1^2 + S_2^2 + S_3^2 = s(s+1)I$$

$s = 1/2$ , the  $2 \times 2$  Pauli-matrices:

$$S_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$s = 1$ , the  $3 \times 3$  Pauli-matrices:

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We will use alternatively  $S_+, S_-, S_3$ , where

$$S_+ := S_1 + iS_2 \quad S_- := S_1 - iS_2$$

which obey

$$[S_3, S_{\pm}] = \pm S_{\pm} \quad [S_+, S_-] = 2S_3 \quad S_- S_+ + S_3(I + S_3) = s(s+1)I$$

(See the bosonic creation & annihilation ops  $b^\dagger, b$  to come soon.)

Let  $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \in \mathbb{R}^3$

and  $S_{\vec{v}} := v_1 S_1 + v_2 S_2 + v_3 S_3 = " \vec{v} \cdot \vec{S}"$ .

Then  $[S_{\vec{v}}, S_{\vec{u}}] = iS_{\vec{v} \times \vec{u}}$ , and  $e^{iS_{\vec{v}}} S_{\vec{u}} e^{-iS_{\vec{v}}} = S_{R_{\vec{v}} \vec{u}}$

where  $R_{\vec{v}} \in SO(3)$  rotates around the axis  $\vec{v}$  with angle  $|\vec{v}|$ .

## The Quantum Heisenberg, or Quantum $XXZ$ model:

$s \in \{1/2, 1, 3/2, \dots\}$  fixed,  $\Lambda = (\mathbb{Z}/L)^d$ ,  $L \in 2\mathbb{N}$ .

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^{2s+1}$$

= multi-linear-forms-over  $\times_{x \in \Lambda} \mathbb{C}^{2s+1}$

$\mathbb{C}$ -Euclidean space, of  $\mathbb{C}$ -dimension  $\dim(\mathcal{H}_\Lambda) = (2s+1)^{|\Lambda|}$ .

**The tensor product is a subtle object!**

Here is the abstract definition:

(turn page)

## Tensor products of Hilbert spaces:

Let  $\mathcal{H}_1, \mathcal{H}_2$  be  $\mathbb{C}$ -Hilbert-spaces, and

$$\mathcal{H}_1 \times \mathcal{H}_2 := \{ \underbrace{\langle \psi_1, \psi_2 \rangle}_{\text{ordereed pairs}} : \psi_1 \in \mathcal{H}_1, \psi_2 \in \mathcal{H}_2 \}$$

**Not a vector space!** It is separately linear in the components:

$$\langle \psi_1 + \psi'_1, \psi_2 \rangle = \langle \psi_1, \psi_2 \rangle + \langle \psi'_1, \psi_2 \rangle$$

$$\langle \psi_1, \psi_2 + \psi'_2 \rangle = \langle \psi_1, \psi_2 \rangle + \langle \psi_1, \psi'_2 \rangle$$

But,  $\langle \psi_1, \psi_2 \rangle + \langle \psi'_1, \psi'_2 \rangle$  is not defined in  $\mathcal{H}_1 \times \mathcal{H}_2$ .

Now, for  $\varphi_1 \in \mathcal{H}_1, \varphi_2 \in \mathcal{H}_2$  let  $\varphi_1 \otimes \varphi_2$  be the *bounded bilinear functional*  $\varphi_1 \otimes \varphi_2 : \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathbb{C}$  acting as

$$\varphi_1 \otimes \varphi_2 \langle \psi_1, \psi_2 \rangle := (\varphi_1, \psi_1)_{\mathcal{H}_1} (\varphi_2, \psi_2)_{\mathcal{H}_2}$$

and

$$(\mathcal{H}_1 \otimes \mathcal{H}_2)^\circ := \left\{ \sum_{k=1}^n c_k \varphi_{k,1} \otimes \varphi_{k,2} : n \in \mathbb{N}, c_k \in \mathbb{C}, \varphi_{k,j} \in \mathcal{H}_j \right\}$$

$$\subseteq \{\text{bdd conj. bilin. functionals over } \mathcal{H}_1 \times \mathcal{H}_2\}$$

endowed with the scalar product sesqui-linearly extended from

$$(\varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2) := (\varphi_1, \psi_1)_{\mathcal{H}_1} (\varphi_2, \psi_2)_{\mathcal{H}_2}$$

$\mathcal{H}_1 \otimes \mathcal{H}_2$  is the closure of  $(\mathcal{H}_1 \otimes \mathcal{H}_2)^\circ$  w.r.t. this scalar product.

Further notation: for  $A \in \mathcal{B}(\mathbb{C}^{2s+1})$  denote

$$A(x) := I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I$$

## Main commutation relations

$$[S_\alpha(x), S_\beta(y)] = i\delta_{x,y}\epsilon_{\alpha,\beta,\gamma}S_\gamma(x)$$

## The $XXZ$ Hamiltonian:

$$\begin{aligned} H_\Lambda &= -\frac{1}{2} \sum_{x \sim y \in \Lambda} (S_1(x)S_1(y) + S_2(x)S_2(y) + uS_3(x)S_3(y)) - h \sum_{x \in \Lambda} S_3(x) \\ &= -\frac{1}{2} \sum_{x \sim y \in \Lambda} (S_+(x)S_-(y) + uS_3(x)S_3(y)) - h \sum_{x \in \Lambda} S_3(x) \end{aligned}$$

Note: Let  $U := \exp\{i\pi \sum_x \text{even } S_3(x)\}$ . Then

$$UH_\Lambda U^* = \frac{1}{2} \sum_{x \sim y \in \Lambda} (S_1(x)S_1(y) + S_2(x)S_2(y) - uS_3(x)S_3(y)) - h \sum_{x \in \Lambda} S_3(x)$$

Hence, nomenclature:

- $u > 0$ : **ferromagnetic**;  $u = +1$ : isotropic
- $u < 0$ : **antiferromagnetic**;  $u = -1$ : isotropic
- $u = 0$ : the  $XY$  model (better said, the  $XX$  model)

**Internal symmetries:** Let, for  $\alpha = 1, 2, 3$ ,

$$\mathbb{S}_\alpha := \sum_{x \in \Lambda} S_\alpha(x)$$

Then

$$(\forall u)(\forall h) : \quad [\mathbb{S}_3, H] = 0, \quad e^{i\theta \mathbb{S}_3} H e^{-i\theta \mathbb{S}_3} = H$$

$$(\forall u)(h = 0) : \quad e^{i\pi \mathbb{S}_1} H e^{-i\pi \mathbb{S}_1} = H$$

$$(|u| = 1)(h = 0) : \quad [\mathbb{S}_{\vec{v}}, H] = 0, \quad e^{i\mathbb{S}_{\vec{v}}} H e^{-i\mathbb{S}_{\vec{v}}} = H$$

Altogether, the internal symmetries of  $H_\Lambda$  are:

- $h \neq 0$ :  $U(1)$
- $h = 0, |u| \neq 1$ :  $U(1) \text{ & } \mathbb{Z}_2$
- $h = 0, |u| = 1$ :  $SU(2)$

We are primarily interested in the breakdown of the continuous symmetry  $U(1)$  or  $SU(2)$ .

The LRO parameter: :

$$\begin{aligned} r^2(\beta) &:= \lim_{\Lambda \rightarrow \mathbb{Z}^d} |\Lambda|^{-1} \sum_{x \in \Lambda} \langle S_1(0)S_1(x) + S_2(0)S_2(x) \rangle_{\Lambda, \beta} \\ &= \lim_{\Lambda \rightarrow \mathbb{Z}^d} |\Lambda|^{-1} \sum_{x \in \Lambda} \langle S_+(0)S_-(x) \rangle_{\Lambda, \beta} \end{aligned}$$

Here, of course,

$$\langle \dots \rangle_\beta := \frac{\text{Tr} (e^{-\beta H} \dots)}{\text{Tr} (e^{-\beta H})}$$

[L Néel (1948)]: "**Néel order**" for AF coupling ( $u < 0$ )  
 [O Penrose, L Onsager (1956)]: **ODLRO** for BEC

## Main Results about the Quantum Heisenberg $XXZ$ Model:

[ND Mermin, H Wagner (1966)]:

- $\beta < \infty$ :  $d = 2 :: r = 0$ .

[FJ Dyson, EH Lieb, B Simon (1978)] with improvements and extensions in [T Kennedy, EH Lieb, BS Shastry (1988)], [K Kubo, T Kishi (1988)], [EJ Neves, JF Perez (1986)], ... :

- $\beta = \infty$ :  $d = 2, s \geq 1, h = 0, u \leq 0 :: r > 0$   
 $d \geq 3, s \geq \frac{1}{2}, h = 0, u \leq 0 :: r > 0$
- $\beta < \infty$ :  $d \geq 3, s \geq \frac{1}{2}, h = 0, u \leq 0 :: \beta > \beta^* \Rightarrow r > 0$ .

**Note:** Only for antiferromagnetic coupling ( $u \leq 0$ ) and zero transversal field ( $h = 0$ ).

[T Kennedy (1985)]:  $u > 1$ : Ising-type behaviour of  $S_3$  prevails.