

Bálint Tóth
(Budapest & Bristol)

Statistical Physics with Continuous Symmetries

II. The $O(\nu)$, or Classical Heisenberg Model

The $O(\nu)$, or Classical Heisenberg Model: $\nu \geq 2$,

$$\Lambda := (\mathbb{Z}/L)^d, \quad \Omega_\Lambda = (S^{\nu-1})^\Lambda, \quad \nu_\Lambda(d\sigma) = \prod_{x \in \Lambda} \underbrace{d\sigma(x)}_{\text{Haar}}$$

The Hamiltonian and the Gibbs measure:

$$H_\Lambda(\underline{\sigma}) := \underbrace{-\frac{1}{2} \sum_{x \sim y \in \Lambda} \sigma(x) \cdot \sigma(y)}_{O(\nu)\text{-symmetry}} - \underbrace{\varepsilon \sum_{x \in \Lambda} \sigma_1(x)}_{\text{symm. breaking}}$$

$$\mu_{\Lambda, \beta, \varepsilon}(d\sigma) := (Z_{\Lambda, \beta, \varepsilon})^{-1} \exp\{-\beta H_\Lambda(\underline{\sigma})\} \prod_{x \in \Lambda} \underbrace{d\sigma(x)}_{\text{Haar}}$$

$$Z_{\Lambda, \beta, \varepsilon} = \int_{(S^{\nu-1})^\Lambda} \exp\{-\beta H_\Lambda(\underline{\sigma})\} \prod_{x \in \Lambda} d\sigma(x)$$

The spontaneous magnetisation:

$$m_\Lambda(\beta, \varepsilon) := \left\langle |\Lambda|^{-1} \sum_{x \in \Lambda} \sigma_1(x) \right\rangle_{\Lambda, \beta, \varepsilon} \stackrel{\text{per. b.c.}}{=} \langle \sigma_1(0) \rangle_{\Lambda, \beta, \varepsilon}$$

$$m(\beta, \varepsilon) := \lim_{\Lambda \nearrow \mathbb{Z}^d} m_\Lambda(\beta, \varepsilon)$$

$$m(\beta) := \lim_{\varepsilon \rightarrow 0} m(\beta, \varepsilon) \quad \begin{cases} = \\ \neq \end{cases} 0 \quad ?$$

The long range order parameter: $\varepsilon = 0$,

$$r_\Lambda(\beta)^2 := \left\langle \left(|\Lambda|^{-1} \sum_{x \in \Lambda} \sigma(x) \right)^2 \right\rangle_{\Lambda, \beta} \stackrel{\text{per. b.c.}}{=} |\Lambda|^{-1} \sum_{x \in \Lambda} \langle \sigma(0) \cdot \sigma(x) \rangle_{\Lambda, \beta}$$

$$r(\beta)^2 := \lim_{\Lambda \nearrow \mathbb{Z}^d} r_\Lambda(\beta)^2 \quad \begin{cases} = \\ \neq \end{cases} 0 \quad ?$$

Theorem 1. [R Griffiths (1966)]

$$r(\beta) > 0 \Rightarrow m(\beta) > 0$$

Theorem 2. [ND Mermin, H Wagner (1966, 1967)]

In dimensions $d = 1, 2$, at any $\beta < \infty$ (that is $T > 0$)

$$m(\beta) = 0$$

Theorem 3. [J Fröhlich, B Simon, T Spencer (1976)]

In dimensions $d \geq 3$ there exists $\beta_ < \infty$ such that for any $\beta > \beta_*$*

$$r(\beta) > 0$$

Conventions on Fourier transforms (valid for the whole course)

The discrete torus and its dual: Let $L_1, \dots, L_d \in \mathbb{N}$ and

$$\Lambda = \{x = (x_1, \dots, x_d) : x_i \in [-L_i/2, L_i/2) \cap \mathbb{Z}, \quad i = 1, \dots, d\}$$

$$\Lambda^* = \left\{p = (p_1, \dots, p_d) : p_i = \frac{2\pi k_i}{L_i}, \quad k \in \Lambda\right\} \subset [-\pi, \pi]^d$$

Later we will assume $L_i \in 2\mathbb{N}$.

The Hilbert spaces $l^2(\Lambda)$ and $l^2(\Lambda^*)$ with the inner products

$$(f, g) = \sum_{x \in \Lambda} \overline{f(x)}g(x) \quad \text{resp.} \quad (f, g)_* = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \overline{f(p)}g(p)$$

The Fourier transform and its inverse:

$$l^2(\Lambda) \ni f \mapsto \hat{f} \in l^2(\Lambda^*) : \quad \hat{f}(p) := \sum_{x \in \Lambda} e^{ip \cdot x} f(x)$$

$$l^2(\Lambda^*) \ni f \mapsto \check{f} \in l^2(\Lambda) : \quad \check{f}(x) := \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} e^{-ip \cdot x} f(p)$$

where $p \cdot x = \sum_{i=1}^d p_i x_i$

It is easy to check that the Fourier transforms are unitary between $l^2(\Lambda)$ and $l^2(\Lambda^*)$ and are inverse of one-another

$$f, g \in l^2(\Lambda) : \quad (\hat{f}, \hat{g})_* = (f, g) \quad \check{\check{f}} = f$$

$$f, g \in l^2(\Lambda^*) : \quad (\check{f}, \check{g}) = (f, g)_* \quad \hat{\hat{f}} = f$$

The Fourier transform diagonalizes (jointly) Töplitz operators:

Let

$$A = (A_{x,y})_{x,y \in \Lambda} \quad A_{x,y} = a(x - y)$$

Then

$$\widehat{Af}(p) = \widehat{a}(p)\widehat{f}(p)$$

The lattice Laplacian:

$$\Delta = (\Delta_{x,y})_{x,y \in \Lambda}, \quad \Delta_{x,y} := \begin{cases} -2d & \text{if } |x - y| = 0 \\ 1 & \text{if } |x - y| = 1 \\ 0 & \text{if } |x - y| > 1 \end{cases}$$

and its Fourier transform: $D : [-\pi, \pi]^d \rightarrow \mathbb{R}_+$

$$D(p) = 2 \sum_{i=1}^d (1 - \cos p_i), \quad D(p) \asymp |p|^2 \text{ as } |p| \ll 1$$

I will assume basic knowledge about the Fourier transform.

Correlation functions (second order):

The (finite volume) correlation function and its FT: $i, j = 1, \dots, \nu$

$$c_{i,j}^\Lambda(x) := \langle \sigma_i(0)\sigma_j(x) \rangle_{\Lambda, \beta, \varepsilon} = \langle \sigma_i(y)\sigma_j(y+x) \rangle_{\Lambda, \beta, \varepsilon}$$

$$\hat{c}_{i,j}^\Lambda(p) := \sum_{x \in \Lambda} e^{ip \cdot x} c_{i,j}^\Lambda(x)$$

As we will prove *uniform-in-* Λ estimates for the correlations, I will drop the superscript Λ . If no danger of confusion explicit notation of dependence on β and ε will also be dropped.

Note:

- For any Λ and $p \in \Lambda^*$, the matrix $(\hat{c}_{i,j}^\Lambda(p))_{i,j=1}^\nu$ is *positive semi-definite*.

- (Here $\varepsilon = 0$) $r_\Lambda(\beta) = \frac{1}{|\Lambda|} \sum_{i=1}^\nu \hat{c}_{i,i}^\Lambda(0)$

Proof of Theorem 2 [Mermin-Wagner]

[following [J Fröhlich, T Spencer (1981)]]

Assume $\nu = 2$ and write $\sigma = (\sigma_1, \sigma_2) = (\cos \theta, \sin \theta)$, $\theta \in [0, 2\pi)$.

The general case, $\nu \geq 2$, is notationally and computationally somewhat more involved, but conceptually similar.

$$H_\Lambda(\underline{\theta}) := -\frac{1}{2} \sum_{x \sim y \in \Lambda} \cos(\theta(x) - \theta(y))) - \varepsilon \sum_{x \in \Lambda} \cos \theta(x)$$

$$\mu_{\Lambda, \beta, \varepsilon}(d\underline{\theta}) := (Z_{\Lambda, \beta, \varepsilon})^{-1} \exp\{-\beta H_\Lambda(\underline{\theta})\} \prod_{x \in \Lambda} d\theta(x)$$

$$Z_{\Lambda, \beta, \varepsilon} = \int_{[0, 2\pi)^\Lambda} \exp\{-\beta H_\Lambda(\theta)\} \prod_{x \in \Lambda} d\theta(x)$$

$$\sigma_1(x) := \cos \theta(x),$$

$$\hat{\sigma}_1(p) := \sum_{x \in \Lambda} e^{ip \cdot x} \sigma_1(x)$$

$$\sigma_2(x) := \sin \theta(x),$$

$$\hat{\sigma}_2(p) := \sum_{x \in \Lambda} e^{ip \cdot x} \sigma_2(x)$$

$$\partial(x) := \frac{\partial}{\partial \theta(x)}$$

$$\hat{\partial}(p) := \sum_{x \in \Lambda} e^{ip \cdot x} \partial_x$$

An **integration-by-parts** formula: for any observable $\underline{\theta} \mapsto A(\underline{\theta})$,

$$\begin{aligned} \langle (\partial(x)A) \rangle_{\Lambda, \beta, \varepsilon} &= \frac{1}{Z_{\Lambda, \beta, \varepsilon}} \int_{[0, \pi)^{\Lambda}} \frac{\partial A(\underline{\theta})}{\partial \theta(x)} e^{-\beta H_{\Lambda}(\underline{\theta})} \prod_{x \in \Lambda} d\theta(x) \\ &= \frac{\beta}{Z_{\Lambda, \beta, \varepsilon}} \int_{[0, \pi)^{\Lambda}} A(\underline{\theta}) \frac{\partial H_{\Lambda}(\underline{\theta})}{\partial \theta(x)} e^{-\beta H_{\Lambda}(\underline{\theta})} \prod_{x \in \Lambda} d\theta(x) \\ &= \beta \langle A(\partial(x)H_{\Lambda}) \rangle_{\Lambda, \beta, \varepsilon} \end{aligned}$$

and hence

$$\langle (\widehat{\partial}(p)A) \rangle_{\Lambda, \beta, \varepsilon} = \beta \langle A (\widehat{\partial}(p)H_\Lambda) \rangle_{\Lambda, \beta, \varepsilon}$$

Note

$$(\widehat{\partial}(p)\widehat{\sigma}_2(-p)) = \sum_{x,y \in \Lambda} e^{ip \cdot (x-y)} \partial_x \sin \theta(y) = \sum_{x \in \Lambda} \sigma_1(x) = \widehat{\sigma}_1(0)$$

Putting these together:

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$$\begin{aligned}
m_\Lambda(\beta, \varepsilon)^2 &= \frac{1}{|\Lambda|^2} \left\langle \left(\hat{\partial}(p) \hat{\sigma}_2(-p) \right) \right\rangle_{\Lambda, \beta, \varepsilon}^2 \\
&= \frac{\beta^2}{|\Lambda|^2} \left\langle \hat{\sigma}_2(-p) \left(\hat{\partial}(p) H_\Lambda \right) \right\rangle_{\Lambda, \beta, \varepsilon}^2 \\
&\leq \frac{\beta^2}{|\Lambda|^2} \left\langle |\hat{\sigma}_2(p)|^2 \right\rangle_{\Lambda, \beta, \varepsilon} \left\langle |\hat{\partial}(p) H_\Lambda|^2 \right\rangle_{\Lambda, \beta, \varepsilon} \tag{1}
\end{aligned}$$

$$\begin{aligned}
\langle \hat{\sigma}_2(p) \hat{\sigma}_2(-p) \rangle_{\Lambda, \beta, \varepsilon} &= \sum_{x, y \in \Lambda} e^{ip \cdot (x - y)} \langle \sigma_2(x) \sigma_2(y) \rangle_{\Lambda, \beta, \varepsilon} \\
&= |\Lambda| \sum_{x \in \Lambda} e^{ip \cdot (x)} \langle \sigma_2(0) \sigma_2(x) \rangle_{\Lambda, \beta, \varepsilon} \\
&= |\Lambda| \hat{c}_{2,2}^\Lambda(p) \tag{2}
\end{aligned}$$

$$\begin{aligned}
\beta \langle (\widehat{\partial}(p)H_\Lambda) (\widehat{\partial}(-p)H_\Lambda) \rangle_{\Lambda,\beta,\varepsilon} &= \langle \widehat{\partial}(p)\widehat{\partial}(-p)H_\Lambda \rangle_{\Lambda,\beta,\varepsilon} \\
&= \sum_{x,y \in \Lambda} e^{ip \cdot (x-y)} \langle \partial_{x,y}^2 H_\Lambda \rangle_{\Lambda,\beta,\varepsilon} \\
&= \sum_{x \sim y \in \Lambda} (1 - e^{ip \cdot (x-y)}) \langle \cos(\theta(x) - \theta(y)) \rangle_{\Lambda,\beta,\varepsilon} \\
&\quad + \varepsilon \sum_{x \in \Lambda} \langle \cos \theta(x) \rangle_{\Lambda,\beta,\varepsilon} \\
&\leq |\Lambda| (D(p) + \varepsilon m_\Lambda(\beta, \varepsilon))
\end{aligned} \tag{3}$$

Finally, (1), (2) and (3) yield

$$\widehat{c}_{2,2}^\Lambda(p) \geq \frac{m_\Lambda(\beta, \varepsilon)^2}{\beta (D(p) + \varepsilon m_\Lambda(\beta, \varepsilon))}$$

and hence

(turn page)

$$1 \geq c_{2,2}^{\Lambda}(0) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \widehat{c}_{2,2}^{\Lambda}(p) \geq \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \frac{m_{\Lambda}(\beta, \varepsilon)^2}{\beta(D(p) + \varepsilon m_{\Lambda}(\beta, \varepsilon))}$$

uniformly as $\Lambda \nearrow \mathbb{Z}^d$.

In the thermodynamic limit,

$$\frac{1}{\beta(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{m(\beta, \varepsilon)^2}{D(p) + \varepsilon m(\beta, \varepsilon)} dp \leq 1$$

Letting $\varepsilon \rightarrow 0$,

$$m(\beta)^2 \leq \beta(2\pi)^d \left(\int_{[-\pi, \pi]^d} D(p)^{-1} dp \right)^{-1} \underset{d=2}{\asymp} 0$$

\square Mermin-Wagner

Proof of Theorem 3 [Fröhlich-Simon-Spencer] (here $\varepsilon = 0$)

The (finite volume) correlation function and its FT: $i, j = 1, \dots, \nu$

$$c_{i,j}^\Lambda(x) := \langle \sigma_i(0)\sigma_j(x) \rangle_{\Lambda,\beta} = \langle \sigma_i(y)\sigma_j(y+x) \rangle_{\Lambda,\beta}$$

$$\hat{c}_{i,j}^\Lambda(p) := \sum_{x \in \Lambda} e^{ip \cdot x} c_{i,j}^{\Lambda,\beta}(x)$$

As we will prove *uniform-in-* Λ estimates for the correlations, I will drop the superscript Λ . If no danger of confusion the superscript β will also be dropped.

Note:

- For any Λ and $p \in \Lambda^*$, the matrix $(\hat{c}_{i,j}^\Lambda(p))_{i,j=1}^\nu$ is *positive semi-definite*.
- $r_\Lambda(\beta) = \frac{1}{|\Lambda|} \sum_{i=1}^\nu \hat{c}_{i,i}^\Lambda(0)$

Main stations of the proof:

(A) Infrared Bound (IRB) on the correlation function:

For all $p \in \Lambda^* \setminus \{0\}$ the matrix

$$\left(\frac{1}{\beta D(p)} \delta_{i,j} - \hat{c}_{i,j}(p) \right)_{i,j=1}^\nu \quad (\text{IRB})$$

is positive definite. **This is the heart of the proof.**

(B) Sum rule – a straightforward identity:

$$\begin{aligned} 1 &= \langle \sigma(0) \cdot \sigma(0) \rangle_\Lambda = \sum_{i=1}^\nu c_{i,i}(0) \stackrel{\text{i.F.t.}}{=} \frac{1}{|\Lambda|} \sum_{i=1}^\nu \sum_{p \in \Lambda^*} \hat{c}_{i,i}(p) \\ &= \frac{1}{|\Lambda|} \sum_{i=1}^\nu \hat{c}_{i,i}(0) + \frac{1}{|\Lambda|} \sum_{i=1}^\nu \sum_{p \in \Lambda^* \setminus \{0\}} \hat{c}_{i,i}(p) \end{aligned} \quad (\text{SR})$$

(C) Conclusion:

$$\begin{aligned}
 r_{\Lambda}(\beta) & \stackrel{\text{FT}}{=} \frac{1}{|\Lambda|} \sum_{i=1}^{\nu} \hat{c}_{i,i}(0) \stackrel{(\text{SR})}{=} 1 - \frac{1}{|\Lambda|} \sum_{i=1}^{\nu} \sum_{p \in \Lambda^* \setminus \{0\}} \hat{c}_{i,i}(p) \\
 & \stackrel{(\text{IRB})}{\geq} 1 - \frac{\nu}{\beta |\Lambda|} \sum_{p \in \Lambda^* \setminus \{0\}} \frac{1}{D(p)} \\
 & \xrightarrow[\text{Riemann}]{\Lambda \nearrow \mathbb{Z}^d} 1 - \frac{\nu}{\beta} \underbrace{\frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} D(p)^{-1} dp}_{I_d < \infty \text{ for } d \geq 3}
 \end{aligned}$$

Hence

$$r(\beta) \geq 1 - \frac{\nu}{\beta} I_d.$$

Proof of the **IRB** [Infrared Bound]:

We will use the following equivalent forms of the Hamiltonian

$$\begin{aligned} H_{\Lambda}(\underline{\sigma}) &= -\frac{1}{2} \sum_{x \sim y \in \Lambda} \underline{\sigma}(x) \cdot \underline{\sigma}(y) \\ &= -\frac{1}{2} \sum_{x, y \in \Lambda} \underline{\sigma}(x) \cdot \Delta_{x,y} \underline{\sigma}(y) \quad \cancel{-d|\Lambda|} \\ &= -\frac{1}{2} \underline{\sigma} \cdot \Delta \underline{\sigma} \quad \cancel{-d|\Lambda|} \\ &= \frac{1}{4} \sum_{x \sim y \in \Lambda} |\underline{\sigma}(x) - \underline{\sigma}(y)|^2 \quad \cancel{-d|\Lambda|} \end{aligned}$$

Let $\underline{v} \in (\mathbb{R}^\nu)^\Lambda$ and define the \underline{v} -dependent partition function

$$\begin{aligned} Z_{\Lambda, \beta}(\underline{v}) &:= \int_{\Omega_\Lambda} e^{\frac{\beta}{2}(\underline{\sigma} + \underline{v}) \cdot \Delta(\underline{\sigma} + \underline{v})} \prod_{x \in \Lambda} d\sigma(x) \\ &= e^{\frac{\beta}{2}\underline{v} \cdot \Delta\underline{v}} Z_{\Lambda, \beta}(\underline{0}) \left\langle e^{\beta\underline{v} \cdot \Delta\underline{\sigma}} \right\rangle_{\Lambda, \beta} \end{aligned}$$

This is actually an exponential moment generating function.

Note:

$$Z_{\Lambda, \beta}(\underline{0}) = Z_{\Lambda, \beta}$$

Theorem 4. [Gaussian Domination and Infrared Bound]

For any even Λ , $\beta < \infty$ and $\underline{v} \in (\mathbb{R}^\nu)^\Lambda$, the following are true

$$(i) \quad Z_{\Lambda, \beta}(\underline{v}) \leq Z_{\Lambda, \beta}(\underline{0})$$

(ii) [Gaussian Domination]

$$\left\langle e^{\underline{v} \cdot \Delta \underline{\sigma}} \right\rangle_{\Lambda, \beta} \leq e^{-\frac{1}{2\beta} \underline{v} \cdot \Delta \underline{v}} \quad (\text{GD})$$

$$(iii) \quad \sum_{i,j=1}^{\nu} \sum_{s,t,x,y \in \Lambda} v_i(s) \Delta_{s,x} \left\langle \sigma_i(x) \sigma_j(y) \right\rangle_{\Lambda, \beta} \Delta_{y,t} v_j(t) \leq -\frac{1}{\beta} \underline{v} \cdot \Delta \underline{v}$$

(iv) [Infrared Bound]: $\forall p \in \Lambda^* \setminus \{0\}$: the matrix

$$\left(\frac{1}{\beta D(p)} \delta_{i,j} - \hat{c}_{i,j}(p) \right)_{i,j=1}^{\nu} \geq 0 \quad (\text{IRB})$$

is positive semi-definite

Proof of Theorem 4 [Gaussian Domination]:
The following equivalences/implications hold

$$(i) \underset{\text{strfwd}}{\Leftrightarrow} (ii) \underset{\text{expansion}}{\Rightarrow} (iii) \underset{\text{FT}}{\Leftrightarrow} (iv)$$

We will prove (i).

Proposition. [Reflection Positivity - classical case]

Let (Ω, μ) be a finite measure space and

$$A, B, C_1, \dots, C_l, D_1, \dots, D_l : \Omega \rightarrow \mathbb{C}$$

bounded measurable functions. The following inequality holds

$$\left| \int_{\Omega \times \Omega} e^{A(s) + B(t) - \frac{1}{2} \sum_{k=1}^l (C_k(s) - D_k(t))^2} d\mu(s) d\mu(t) \right|^2 \leq \quad (\textbf{RP})$$

$$\int_{\Omega \times \Omega} e^{A(s) + \bar{A}(t) - \frac{1}{2} \sum_{k=1}^l (C_k(s) - \bar{C}_k(t))^2} d\mu(s) d\mu(t)$$

$$\times \int_{\Omega \times \Omega} e^{\bar{B}(s) + B(t) - \frac{1}{2} \sum_{k=1}^l (\bar{D}_k(s) - D_k(t))^2} d\mu(s) d\mu(t)$$

Remarks: (1) It is not just a plain Schwarz.

(2) In the proof of Thm 3 we will only need the real case, but with view on the QHM it is instructive to see the complex setting.

Proof of (RP) [Reflection Positivity Lemma]: I will prove it for $l = 1$. For general $l \geq 1$ only the notation gets more complicated.

$$\begin{aligned}
& \left| \int_{\Omega \times \Omega} d\mu(s) d\mu(t) e^{A(s)+B(t)-\frac{1}{2}(C(s)-D(t))^2} \right|^2 \\
& \stackrel{1}{=} \left| \int_{\Omega \times \Omega} d\mu(s) d\mu(t) e^{A(s)+B(t)} \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} e^{i\xi(C(s)-D(t))} \right|^2 \\
& \stackrel{2}{=} \left| \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \left(\int_{\Omega} d\mu(s) e^{A(s)+i\xi C(s)} \right) \left(\int_{\Omega} d\mu(t) e^{B(t)-i\xi D(t)} \right) \right|^2 \\
& \stackrel{3}{\leq} \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{\Omega} d\mu(s) e^{A(s)+i\xi C(s)} \int_{\Omega} d\mu(t) e^{\overline{A}(t)-i\xi \overline{C}(t)} \times \\
& \quad \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{\Omega} d\mu(s) e^{\overline{B}(s)+i\xi \overline{D}(s)} \int_{\Omega} d\mu(t) e^{B(t)-i\xi D(t)} \\
& \quad \dots
\end{aligned}$$

$$\stackrel{4}{=} \int_{\Omega \times \Omega} d\mu(s) d\mu(t) e^{A(s) + \bar{A}(t)} \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} e^{i\xi(C(s) - \bar{C}(t))} \times$$

$$\int_{\Omega \times \Omega} d\mu(s) d\mu(t) e^{\bar{B}(s) + B(t)} \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} e^{i\xi(\bar{D}(s) - D(t))}$$

$$\stackrel{5}{=} \int_{\Omega \times \Omega} d\mu(s) d\mu(t) e^{A(s) + \bar{A}(t) - \frac{1}{2}(C(s) - \bar{C}(t))^2} \times$$

$$\int_{\Omega \times \Omega} d\mu(s) d\mu(t) e^{\bar{B}(s) + B(t) - \frac{1}{2}(\bar{D}(s) - D(t))^2}$$

(1) & (5): use $e^{-z^2/2} = \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} e^{i\xi z}, \quad z \in \mathbb{C}$

(2) & (4): straightforward rearrangement

(3) Schwarz.

$\square(RP)$

Back to the proof of Gaussian Domination:

We assume that the discrete torus Λ is of even side-lengths. Divide Λ in two symmetric halves by a hyperplane intersecting (cutting) only edges

$$\Lambda = \Lambda_{\text{right}} \cup \Lambda_{\text{left}}$$

and define the natural reflection through the dividing hyperplane

$$R : \Lambda \rightarrow \Lambda$$

We apply the Reflection Positivity Lemma in the following setting

$$\Omega = (S^{\nu-1})^{\Lambda_{\text{right}}} = (S^{\nu-1})^{\Lambda_{\text{left}}}$$

$$s = (\sigma(x))_{x \in \Lambda_{\text{right}}}, \quad t = (\sigma(x))_{x \in \Lambda_{\text{left}}}$$

$$d\mu(s) = \prod_{x \in \Lambda_{\text{right}}} d\sigma(x), \quad d\mu(t) = \prod_{x \in \Lambda_{\text{left}}} d\sigma(x)$$

$$A(s) = -\frac{\beta}{4} \sum_{x \sim y \in \Lambda_{\text{right}}} (\sigma(x) + v(x) - \sigma(y) - v(y))^2$$

$$B(t) = -\frac{\beta}{4} \sum_{x \sim y \in \Lambda_{\text{left}}} (\sigma(x) + v(x) - \sigma(y) - v(y))^2$$

$$k = (x \sim y, i) : \quad x \in \Lambda_{\text{right}}, \quad y \in \Lambda_{\text{left}}, \quad i = 1, \dots, \nu$$

$$(C_k(s) - D_k(t))^2 = \beta (\sigma_i(x) + v_i(x) - \sigma_i(y) - v_i(y))^2$$

For a vector field $\underline{v} \in (\mathbb{R}^\nu)^\Lambda$ let $\underline{v}_r, \underline{v}_l \in (\mathbb{R}^\nu)^\Lambda$ be the following

$$v_r(x) := \begin{cases} v(x) & \text{if } x \in \Lambda_{\text{right}}, \\ v(Rx) & \text{if } x \in \Lambda_{\text{left}} \end{cases}, \quad v_l(x) = \begin{cases} v(x) & \text{if } x \in \Lambda_{\text{left}} \\ v(Rx) & \text{if } x \in \Lambda_{\text{right}} \end{cases}$$

In this setting (RP) yields:

$$Z_\Lambda^2(\underline{v}) \leq Z_\Lambda(\underline{v}_r)Z_\Lambda(\underline{v}_l) \tag{4}$$

Since $\lim_{|\underline{v}| \rightarrow \infty} Z_\Lambda(\underline{v}) = 0$, $\operatorname{argmax} Z_\Lambda(\cdot) \subset (\mathbb{R}^\nu)^\Lambda$ is finite

Let $\underline{v}^* \in \operatorname{argmax} Z_\Lambda(\cdot)$ be such that

$$\gamma(v^*) := \{(x, y) \text{ edge in } \Lambda : v^*(x) \neq v^*(y)\}$$

is *minimal*.

If $\gamma(v^*) = \emptyset$, then $x \mapsto v^*(x)$ is constant and we are done.

Assume that $\gamma(v^*) \neq \emptyset$ and realize the reflection of the previous argument through a hyperplane cutting at least one edge in $\gamma(v^*)$. From (4) we obtain that $v_r^*, v_l^* \in \operatorname{argmax} Z_\Lambda(\cdot)$.

But, from the choice of $v^* \in \operatorname{argmax} Z_\Lambda(\cdot)$ and the reflection plane it clearly follows that

$$\begin{aligned} \min\{\#\gamma(v_r^*), \#\gamma(v_l^*)\} &= 2 \min\{\#\gamma(v^*) \cap \Lambda_r, \#\gamma(v^*) \cap \Lambda_l\} \\ &< \#\gamma(v^*) \end{aligned}$$

which contradicts the choice of $v^* \in \operatorname{argmax} Z_\Lambda(\cdot)$.

□ Thm 4

□ Fröhlich-Simon-Spencer