Advanced Topics in Numerical Analysis

Homework 2

Francesco Tudisco

GSSI

Policies

These homeworks are not mandatory for this course but you are encouraged to try to solve them and any work in this direction will be highly appreciated. You may discuss your homework problems freely with other students, but please refrain from looking at their code or writeups. Please submit your solutions using a pdf file generated using LaTeX.

Questions

Let A be the adjacency matrix of a graph $G = (V, E)$ without self-loops ($A_{ii} = 0$ for all i) and let $L = \mathrm{Diag}(A\mathbb{1}) - A$ be its Laplacian matrix. A subset $S \subseteq V$ is connected if for any two nodes $i, j \in S$ there exists a path (sequence of adjacent edges) that connects i and j. A connected component is a connected set $S \subseteq V$ that is not contained in a larger connected set.

1. Show that $\ker L = \mathrm{span}\{\mathbb{1}_{S_1}, \dots, \mathbb{1}_{S_k}\}$ where $\{S_i\}$ are the connected components of G
2. Prove the formula $x^\top Lx = \frac{1}{2}\sum_{ij} A_{ij}(x_i - x_j)^2$

For positive numbers $\mu_i > 0$, let $\mathrm{cut}(S) = \sum_{i\in S, j\notin S} A_{ij}$, $\mu(S) = \sum_{i\in S}\mu_i$ and $\gamma(S) = \mathrm{cut}(S)\max\{\mu(S)^{-1}, \mu(\overline{S})^{-1}\}$, where $\overline{S} = V \setminus S$ is the complement of S in V. Let $D_\mu = \mathrm{Diag}(\mu_1, \dots, \mu_n)$ and for $S \subseteq V$ let $U_S = \mathrm{span}\{\mathbb{1}_S, \mathbb{1}_{\overline{S}}\}$. Show that

3. For any $u \in U_S$ it holds
$$\frac{u^* L u}{u^* D_\mu u} \leq 2\gamma(S)$$
4. Deduce one direction of the Cheeger inequality: $\min_S \gamma(S) \geq \lambda_2(L_\mu)/2$ where $L_\mu = D_\mu^{-1/2} L D_\mu^{-1}$
5. For a function $f : V \to \mathbb{R}^k$ let $H(f) = \sum_{ij} A_{ij}\|f(i) - f(j)\|_2^2$ and let M_f be the matrix with rows $(M_f)_{i,:} = f(i)$. Show that
$$\min\{H(f) : M_f^* M_f = I\} = \lambda_1(L) + \cdots + \lambda_k(L)$$
and that the minimizer is the function f such that $\mathrm{ran}\, M_f$ = eigenspace corresponding to the eigenvalues $\{\lambda_1(L), \dots, \lambda_k(L)\}$. Can we claim that the same minimum is obtained if we look for any M_f such that $\mathrm{rank}\, M_f = k$ + some suitable normalization?
6. Show that A is irreducible if and only if the graph with adjacency matrix $|A|$ is strongly connected (i.e. made by only one connected component)