Advanced Topics in Numerical Analysis

GSSI

Homework 1

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Policies

These homeworks are not mandatory for this course but you are encouraged to try to solve them and any work in this direction will be highly appreciated. You may discuss your homework problems freely with other students, but please refrain from looking at their code or writeups. Please submit your solutions using a pdf file generated using LaTeX.

Questions

1. Let
$$A = \begin{bmatrix} 1 & 1 \\ 0 & -(2\varepsilon)^{-1} \end{bmatrix}.$$
Show that $\kappa_2(A) = O(1/\varepsilon)$

2. Use Schur's theorem to show that for any normal matrix A there exists a unitary matrix $U = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}$ such that $A = \lambda_1 u_1 u_1^* + \cdots + \lambda_n u_n u_n^*$

3. Use Schur's theorem to show that the set of diagonalizable matrices is dense in $\mathbb{C}^{n\times n}$

4. Let $\|A\|_p := \max_{\|x\|_p=1} \|Ax\|_p$.
Show that $\|AB\|_p \le \|A\|_p \|B\|_p$ and that $\|A\|_2 = \max\{|\lambda| : \lambda \in \sigma(A^*A)\}$

5. Let A and B be Hermtian matrices and assume their eigenvalues are sorted in ascending order: $\lambda_1(M) \le \cdots \le \lambda_n(M)$. Prove the following statements:
   (a) $\lambda_1(A) + \cdots + \lambda_k(A) = \min\{\mathrm{Tr}(Y^*AY) : Y^*Y = I_k\}$, I_k= identity of size k.
   (b) $\sum_{i=1}^k \lambda_i(A+B) \ge \sum_{i=1}^k \lambda_i(A) + \lambda_i(B)$.
   (c) $\min_{x\perp U_k} r_A(x) = \lambda_{k+1}(A)$, where U_k is the eigenspace of the first k eigenvalues of A.
   (d) $\lambda_k(A) + \lambda_1(B) \le \lambda_k(A+B) \le \lambda_k(A) + \lambda_n(B)$.
   (e) $\max_k |\lambda_k(A) - \lambda_k(B)| \le \|A - B\|_2$.
   (f) (Weyl's inequalities) Sort the eigenvalues in desceding order: $\lambda_1(M) \ge \cdots \ge \lambda_n(M)$. Then:
   $$\lambda_{i+j-1}(A+B) \le \lambda_i(A) + \lambda_j(B)$$
   Hint. Use the variational characterization with spaces of dimension $n-i+1$ (for A) and $n-j+1$ (for B) formed by the corresponding right-most eigenvectors and notice that any $(i+j-1)$-dimensional subspace has nontrivial intersection with both such spaces.