Gibbs measures of nonlinear Schrödinger equations as limits of many-body quantum states in dimension $d \leqslant 3$

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The nonlinear Schrödinger equation

Consider the spatial domain $\Lambda = \mathbb{T}^d$ for $d = 1, 2, 3$.

- Study the ***nonlinear Schrödinger equation (NLS)***.

$$\begin{cases} \mathrm{i}\partial_t \phi_t(x) \;=\; \big(-\Delta + \kappa\big)\phi_t(x) + \int \mathrm{d}y\, w(x-y)\, |\phi_t(y)|^2\, \phi_t(x) \\ \phi_0(x) \;=\; \Phi(x) \in H^s(\Lambda)\,. \end{cases}$$

- $\kappa > 0$ and $w \in L^\infty(\Lambda)$ is *positive* or $w = \delta$.
- Sobolev space H^s with norm $\|f\|_{H^s(\Lambda)} \;:=\; \|(1-\Delta)^{s/2} f\|_{L^2(\Lambda)}$.
- Conserved energy

$$H(\phi) \;=\; \int \mathrm{d}x\, \big(|\nabla\phi(x)|^2 + \kappa|\phi(x)|^2\big) + \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, |\phi(x)|^2\, w(x-y)\, |\phi(y)|^2\,.$$

$\rightarrow$ *An infinite-dimensional Hamiltonian system*.

Hamiltonian systems

A general ***Hamiltonian system*** is comprised of the following.

(1) Phase space Γ. We denote its elements by ϕ.

(2) Hamilton (energy) function $H \in C^\infty(\Gamma)$.

(3) Poisson bracket $\{\cdot,\cdot\} : C^\infty(\Gamma) \times C^\infty(\Gamma) \to C^\infty(\Gamma)$ satisfying

- Antisymmetry : $\{f,g\} = -\{g,f\}$.
- Distributivity : $\{f+g,h\} = \{f,h\} + \{g,h\}$.
- Leibniz rule: $\{fg,h\} = \{f,h\}g + f\{g,h\}$.
- Jacobi identity : $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$.

The ***Hamiltonian flow*** $\phi \mapsto \phi_t$ of H on Γ is determined by the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\phi_t) \; = \; \{H,f\}(\phi_t)$$

for $f \in C^\infty(\Gamma)$.

NLS as a Hamiltonian system

(1) Phase space $\Gamma = H^s(\Lambda)$ for some $s \in \mathbb{R}$.

(2) Hamilton function is

$$H(\phi) \;=\; \int \mathrm{d}x \left(|\nabla\phi(x)|^2 + \kappa|\phi(x)|^2\right) + \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,|\phi(x)|^2\,w(x-y)\,|\phi(y)|^2\,.$$

(3) Poisson bracket is

$$\{\phi(x),\bar\phi(y)\} \;=\; \mathrm{i}\delta(x-y)\,,\quad \{\phi(x),\phi(y)\} \;=\; \{\bar\phi(x),\bar\phi(y)\} \;=\; 0\,.$$

Hamiltonian equations of motion are given by the ***nonlinear Schrödinger equation***

$$\mathrm{i}\partial_t\phi_t(x) + (\Delta-\kappa)\phi_t(x) \;=\; \int \mathrm{d}y\,w(x-y)\,|\phi_t(y)|^2\,\phi_t(x)\,.$$

Gibbs measures for the NLS

- The ***Gibbs measure*** $\mathrm{d}\mu$ associated to H is the probability measure on the space of fields $\phi : \Lambda \to \mathbb{C}$

$$\mu(\mathrm{d}\phi) \;:=\; \frac{1}{Z}\mathrm{e}^{-H(\phi)}\,\mathrm{d}\phi\,, \qquad Z \;:=\; \int \mathrm{e}^{-H(\phi)}\,\mathrm{d}\phi\,.$$

  $\mathrm{d}\phi =$ (formally-defined) Lebesgue measure.

- Formally, $\mathrm{d}\mu$ is invariant under flow of NLS:

$$(S_t)_*\,\mu \;=\; \mu \quad \text{for} \quad S_t := \text{ flow map of NLS}\,.$$

- Difficulty: infinite-dimensional Hamiltonian system.

Gibbs measures for the NLS: known results

- **Rigorous construction:** CQFT literature in the 1970-s (Nelson, Glimm-Jaffe, Simon), also Lebowitz-Rose-Speer (1988).
- **Proof of invariance:** Bourgain and Zhidkov (1990s).
- **Application to PDE:** *Obtain low-regularity solutions of NLS μ-almost surely*.
  Recent advances: Bourgain-Bulut, Burq-Tzvetkov, Burq-Thomann-Tzvetkov, Cacciafesta- de Suzzoni, Deng, Genovese-Lucá-Valeri, Nahmod-Oh-Rey-Bellet-Staffilani, Nahmod-Rey-Bellet-Sheffield-Staffilani, Oh-Pocovnicu, Oh-Quastel, Oh-Tzvetkov, Oh-Tzvetkov-Wang, Thomann-Tzvetkov, Tzvetkov, ...

The Wiener measure and classical free field

- Let $H_0(\phi) := \int \mathrm{d}x\,(|\nabla\phi(x)|^2 + \kappa|\phi(x)|^2)$. Define the ***Wiener measure*** $\mathrm{d}\mu_0$

$$\mu_0(\mathrm{d}\phi) \;:=\; \frac{1}{Z_0}\mathrm{e}^{-H_0(\phi)}\,\mathrm{d}\phi\,, \qquad Z_0 \;:=\; \int \mathrm{e}^{-H_0(\phi)}\,\mathrm{d}\phi\,.$$

- Write $a_k := \widehat{\phi}(k)$ and $\mathrm{d}^2 a_k := \mathrm{d}\,\mathrm{Im} a_k\;\mathrm{d}\,\mathrm{Re} a_k$.

$$\mu_0(\mathrm{d}\phi) \;=\; \prod_{k\in\mathbb{Z}^d} \frac{\mathrm{e}^{-c(|k|^2+\kappa)|a_k|^2}\mathrm{d}^2 a_k}{\int \mathrm{e}^{-c(|k|^2+\kappa)|a_k|^2}\mathrm{d}^2 a_k}\,.$$

For $\phi \in \mathrm{supp}\,\mathrm{d}\mu_0$, $(|k|^2+\kappa)^{1/2}\widehat{\phi}(k)$ *has a Gaussian distribution*.

$$\phi \;\equiv\; \phi^\omega \;=\; \sum_{k\in\mathbb{Z}^d} \frac{g_k(\omega)}{(|k|^2+\kappa)^{1/2}}\,\mathrm{e}^{2\pi i k\cdot x}\,,\;\; (g_k) \;=\; \text{i.i.d. complex Gaussians.}$$

$\to$ ***Classical free field***.

- Series converges almost surely in $H^{1-\frac{d}{2}-\varepsilon}(\Lambda)$ since for $s < 1 - \frac{d}{2}$.

$$\mathbb{E}_{\mu_0}\|\phi^\omega\|_{H^s}^2 \;=\; \sum_{k\in\mathbb{Z}^d} (|k|^2+1)^s \frac{\mathbb{E}\big(|g_k|^2\big)}{|k|^2+\kappa} \;\sim_\kappa\; \sum_{k\in\mathbb{Z}^d} (|k|^2+1)^{s-1} \;<\; \infty\,.$$

The classical system and Gibbs measures

- The *classical interaction* is

$$W := \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, |\phi^\omega(x)|^2\, w(x-y)\, |\phi^\omega(y)|^2 .$$

- In $[0,+\infty)$ almost surely if $d=1$ and $w \in L^\infty(\mathbb{T}^1)$ is *pointwise nonnegative*.
- In this case $\mathrm{d}\mu$ is a well-defined probability measure on $H^{1/2-\varepsilon}(\mathbb{T}^1)$ which satisfies

$$\mathrm{d}\mu \ll \mathrm{d}\mu_0 .$$

- For $d=2,3$, W *is infinite almost surely* even if $w \in L^\infty(\mathbb{T}^d)$.

The classical system and Gibbs measures

- Perform a *renormalisation* in the form of ***Wick ordering***. Formally replace W by the *Wick-ordered classical interaction*

$$W^w := \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, \left(|\phi^\omega(x)|^2 - \infty\right) w(x-y) \left(|\phi^\omega(y)|^2 - \infty\right).$$

- Rigorously defined as limit in $\bigcap_{m \geqslant 1} L^m(\mathrm{d}\mu_0)$ of truncations

$$W_{[K]} := \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, \left(|\phi^\omega_{[K]}(x)|^2 - \varrho_K\right) w(x-y) \left(|\phi^\omega_{[K]}(y)|^2 - \varrho_K\right).$$

$$\phi^\omega_{[K]}(x) := \sum_{|k| \leqslant K} \frac{g_k(\omega)}{\sqrt{|k|^2 + \kappa}} \mathrm{e}^{2\pi i k \cdot x}, \quad \varrho_K(x) := \mathbb{E}_{\mu_0} |\phi^\omega_{[K]}(x)|^2 \to \infty.$$

- $W \equiv W^w \geqslant 0$ almost surely if $\hat{w}$ is *pointwise nonnegative*, i.e. w is of *positive type*.

The classical system and Gibbs measures

- **Classical Gibbs state** $\rho(\cdot)$: Given $X \equiv X(\omega)$ a random variable, let

$$\rho(X) := \frac{\int X \, \mathrm{e}^{-W} \, \mathrm{d}\mu_0}{\int \mathrm{e}^{-W} \, \mathrm{d}\mu_0} = \int X \, \mathrm{d}\mu \, .$$

- **The p-particle space**: Denote by

$$\mathfrak{H}^{(p)} := L^2_{\mathrm{sym}}(\Lambda^p)$$

  the elements of $L^2(\Lambda^p)$ which are symmetric in their arguments.
- On $\mathfrak{H}^{(p)}$ define the ***classical p-particle correlation function*** γ_p by its operator kernel

$$\gamma_p(x_1, \dots, x_p; y_1, \dots, y_p) := \rho\big(\overline{\phi^\omega}(y_1) \cdots \overline{\phi^\omega}(y_p) \phi^\omega(x_1) \cdots \phi^\omega(x_p)\big) \, .$$

- For $f \in H^{-1}(\Lambda)$ let $\phi(f) := \langle f, \phi \rangle_{L^2}$, $\bar{\phi}(f) := \langle \phi, f \rangle_{L^2}$.
- Given $f_1, \dots, f_p, g_1, \dots, g_q \in H^{-1}(\Lambda)$, we have

$$\rho\big(\bar{\phi}(g_1) \cdots \bar{\phi}(g_q) \phi(f_1) \cdots \phi(f_p)\big) = \delta_{pq} \big\langle f_1 \otimes \cdots \otimes f_p \, , \gamma_p \, g_1 \otimes \cdots \otimes g_q \big\rangle \, .$$

Derivation of Gibbs measures: informal statement

Formally, ***NLS is a classical limit of many-body quantum theory***.

- On $\mathfrak{H}^{(n)}$ we consider the n*-particle Hamiltonian*

$$H^{(n)} := \sum_{i=1}^{n} (-\Delta_{x_i} + \kappa) + \frac{1}{n} \sum_{1 \leqslant i < j \leqslant n} w(x_i - x_j)\,.$$

- Solve *n-body Schrödinger equation*

$$\mathrm{i}\partial_t \Psi_{n,t} = H^{(n)} \Psi_{n,t}$$

and obtain, as $n \to \infty$

$$\Psi_{n,0} \sim \phi_0^{\otimes n} \quad \text{implies} \quad \Psi_{n,t} \sim \phi_t^{\otimes n}\,.$$

(Hepp (1974), Ginibre-Velo (1979), Spohn (1980), ...).

- **Problem:** Obtain Gibbs measure $\mathrm{d}\mu$ as many-body quantum limit .

The quantum problem

- At temperature $\tau > 0$, equilibrium of $H^{(n)}$ is governed by the *Gibbs state*

$$\frac{1}{Z_\tau^{(n)}} \mathrm{e}^{-H^{(n)}/\tau}, \qquad Z_\tau^{(n)} := \mathrm{Tr}\, \mathrm{e}^{-H^{(n)}/\tau}.$$

- **Goal:** Obtain correlation functions γ_p in limit as $\tau = n \to \infty$.
- Work on the *Bosonic Fock space*

$$\mathcal{F} := \bigoplus_{n \in \mathbb{N}} \mathfrak{H}^{(n)}$$

with *quantum Hamiltonian*

$$H_\tau := \frac{1}{\tau} \bigoplus_{n \in \mathbb{N}} H^{(n)}.$$

- On $\mathcal{F}$ define the *grand canonical ensemble* by

$$P_\tau := \mathrm{e}^{-H_\tau} = \bigoplus_{n \in \mathbb{N}} \mathrm{e}^{-H^{(n)}/\tau}.$$

Second quantisation

- Introduce *quantum fields (operator-valued distributions)* ϕ_τ, ϕ_τ^* on $\mathcal{F}$ with

$$[\phi_\tau(x), \phi_\tau^*(y)] = \frac{1}{\tau}\delta(x-y)\,, \quad [\phi_\tau(x), \phi_\tau(y)] = [\phi_\tau^*(x), \phi_\tau^*(y)] = 0\,.$$

- More precisely, for $f \in L^2(\Lambda)$ define the *creation* and *annihilation operators* $b^*(f)$ and $b(f)$ acting on $\Psi \in \mathcal{F}$ by

$$\big(b^*(f)\Psi\big)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^n f(x_i) \Psi^{(n-1)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\,,$$

$$\big(b(f)\Psi\big)^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \int \mathrm{d}x\, \bar{f}(x)\, \Psi^{(n+1)}(x, x_1, \ldots, x_n)\,.$$

- Take $\phi_\tau(f) := \tau^{-1/2} b(f)\,,\ \phi_\tau^*(f) := \tau^{-1/2} b^*(f)$ and write

$$\phi_\tau^*(f) = \int \mathrm{d}x\, f(x)\, \phi_\tau^*(x)\,, \quad \phi_\tau(f) = \int \mathrm{d}x\, \bar{f}(x)\, \phi_\tau(x)\,.$$

The quantum Gibbs state

- Quantum Gibbs state $\rho_\tau(\cdot)$: Given $\mathcal{A} \in \mathcal{L}(\mathcal{F})$ we define its expectation

$$\rho_\tau(\mathcal{A}) := \frac{\mathrm{Tr}(\mathcal{A} P_\tau)}{\mathrm{Tr}(P_\tau)}.$$

- On $\mathfrak{H}^{(p)}$ define the ***quantum p-particle correlation function*** $\gamma_{\tau,p}$ by its kernel

$$\gamma_{\tau,p}(x_1,\ldots,x_p;y_1,\ldots,y_p) = \rho_\tau\big(\phi_\tau^*(y_1)\cdots\phi_\tau^*(y_p)\phi_\tau(x_1)\cdots\phi_\tau(x_p)\big).$$

- Given $f_1,\ldots,f_p,g_1,\ldots,g_q \in L^2(\Lambda)$ we have

$$\rho_\tau\big(\phi_\tau^*(g_1)\cdots\phi_\tau^*(g_q)\phi_\tau(f_1)\cdots\phi_\tau(f_p)\big) = \delta_{pq}\big\langle f_1\otimes\cdots\otimes f_p\,,\gamma_{\tau,p}\,g_1\otimes\cdots\otimes g_q\big\rangle.$$

Derivation of Gibbs measures: statement of result

Theorem 1: Fröhlich, Knowles, Schlein, S. (CMP, 2017).

(i) Let $d = 1$ and $w \in L^\infty(\mathbb{T}^1)$ be pointwise nonnegative. Then for all $p \in \mathbb{N}$ we have

$$\gamma_{\tau,p} \to \gamma_p \quad \text{as} \quad \tau \to \infty\,.$$

The convergence is in the trace class. ($\|\mathcal{A}\|_{\mathrm{Tr}} := \mathrm{Tr}\,|\mathcal{A}|$).

(ii) Let $d = 2, 3$ and $w \in L^\infty(\mathbb{T}^d)$ be of positive type. The convergence holds in the Hilbert-Schmidt class *after a renormalisation procedure* and with a *slight modification of the grand canonical ensemble* P_τ (needed for technical reasons).

Remark:
Our result applies on $\mathbb{R}^d, d = 1, 2, 3$ if instead of $-\Delta + \kappa$ we consider the one-body Hamiltonian

$$h = -\Delta + \kappa + v$$

for sufficiently confining $v : \mathbb{R}^d \to [0, \infty)$.

Related results

- $1D$ result: previously shown using different techniques by Lewin-Nam-Rougerie (J. Éc. Polytech. Math., 2015).
  In higher dimensions, they consider non local, non translation-invariant interactions.
- Lewin-Nam-Rougerie (J. Math. Phys. 2018) : $1D$ non-periodic problem with subharmonic trapping.
- Lewin-Nam-Rougerie (preprint 2018) : $2D$ problem with translation-invariant interaction without modified Gibbs state.

The high-temperature limit in the free case

Examine the limit $\tau \to \infty$ in the *free case* $w = 0$.

- Define the *rescaled particle number operator* by

$$\mathcal{N}_\tau \;:=\; \frac{1}{\tau} \bigoplus_{n \in \mathbb{N}} n \mathrm{I}_{\mathfrak{H}^{(n)}} \;=\; \int \mathrm{d}x \, \phi_\tau^*(x) \, \phi_\tau(x) \,.$$

- Compare with

$$\mathcal{N} \;:=\; \int \mathrm{d}x \, |\phi^\omega(x)|^2 \,.$$

- We have

$$\rho_\tau(\mathcal{N}_\tau) \;=\; \sum_{k \in \mathbb{Z}^d} \frac{1}{\tau \big(\mathrm{e}^{\frac{|k|^2 + \kappa}{\tau}} - 1\big)} \;\sim\; \begin{cases} 1 & \text{if } d = 1 \\ \log \tau & \text{if } d = 2 \\ \tau^{1/2} & \text{if } d = 3 \,. \end{cases}$$

$\to$ *Need to renormalise when* $d = 2, 3$.

$\to$ $\rho_\tau(\cdot)$ has a *natural cut-off* for $|k| \geqslant \sqrt{\tau}$.

Renormalisation in the quantum problem

Consider the quantum problem for $d = 2, 3$.

- On $\mathcal{F}$ define the *free quantum Hamiltonian*

$$H_{\tau,0} := \frac{1}{\tau} \bigoplus_{n \in \mathbb{N}} H_0^{(n)} ,$$

  where $H_0^{(n)} := \sum_{i=1}^{n} (-\Delta_{x_i} + \kappa)$.

- Given $\mathcal{A} \in \mathcal{L}(\mathcal{F})$ let

$$\rho_{\tau,0}(\mathcal{A}) := \frac{\mathrm{Tr}(\mathcal{A}\,\mathrm{e}^{-H_{\tau,0}})}{\mathrm{Tr}(\mathrm{e}^{-H_{\tau,0}})} .$$

- The *Wick-ordered many-body Hamiltonian* is

$$H_\tau := H_{\tau,0} + W_\tau , \quad \text{for}$$

$$W_\tau := \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, \big(\phi_\tau^*(x)\phi_\tau(x) - \varrho_\tau(x)\big) \, w(x-y) \, \big(\phi_\tau^*(y)\phi_\tau(y) - \varrho_\tau(y)\big) .$$

- $\varrho_\tau(x) := \rho_{\tau,0}\big(\phi_\tau^*(x)\phi_\tau(x)\big) = \varrho_\tau(0) \to \infty$ as $\tau \to \infty$.

Proof of Theorem 1: perturbative expansion

- **Example**: Consider the *classical partition function*

$$A(z) := \int \mathrm{e}^{-zW}\,\mathrm{d}\mu_0$$

and the *quantum partition function*

$$A_\tau(z) \;=\; \frac{\mathrm{Tr}\left(\mathrm{e}^{-\eta H_{\tau,0}}\,\mathrm{e}^{-(1-2\eta)H_{\tau,0}-zW_\tau}\,\mathrm{e}^{-\eta H_{\tau,0}}\right)}{\mathrm{Tr}(\mathrm{e}^{-H_{\tau,0}})}\,,\qquad \eta \in [0,1/4]\,.$$

Modification by η: In $2D$ and $3D$ we replace P_τ by $P_\tau^\eta \;:=\; \mathrm{e}^{-\eta H_{\tau,0}}\mathrm{e}^{-(1-2\eta)H_{\tau,0}-W_\tau}\mathrm{e}^{-\eta H_{\tau,0}}\,,\;\; \eta \neq 0\,.$

- Our goal is to prove that

$$\lim_{\tau\to\infty} A_\tau(z) \;=\; A(z) \quad \text{for } \operatorname{Re} z > 0\,.$$

- **Problem:** The series expansions

$$A(z) \;=\; \sum_{m=0}^{\infty} a_m z^m\,,\quad A_\tau(z) \;=\; \sum_{m=0}^{\infty} a_{\tau,m} z^m$$

have *radius of convergence zero*.

Idea of proof: Borel summation

- Recover $A(z), A_\tau(z)$ from their coefficients by ***Borel summation***.
- Given a formal power series

$$\mathcal{A}(z) = \sum_{m \geqslant 0} \alpha_m z^m$$

its *Borel transform* is

$$\mathcal{B}(z) := \sum_{m \geqslant 0} \frac{\alpha_m}{m!} z^m .$$

Formally we have

$$\mathcal{A}(z) = \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-t} \, \mathcal{B}(tz) .$$

Application of Borel summation

For fixed $M \in \mathbb{N}$ write

$$A(z) = \sum_{m=0}^{M-1} a_m z^m + R_M(z)\,, \quad A_\tau(z) = \sum_{m=0}^{M-1} a_{\tau,m} z^m + R_{\tau,M}(z)\,.$$

By a result of Sokal (1980), it suffices to prove the following.

(i) The *explicit terms* satisfy

$$|a_m| + |a_{\tau,m}| \leqslant C^m m!\,,$$

and the *remainder terms* satisfy

$$|R_M(z)| + |R_{\tau,M}(z)| \leqslant C^M M! |z|^M\,.$$

The bound on $R_{\tau,M}$ requires $\eta \neq 0$ in $2D$ and $3D$.

(ii) The quantum coefficients converge to the classical coefficients, i.e.

$$\lim_{\tau \to \infty} a_{\tau,m} = a_m\,.$$

Estimating the explicit terms in the quantum setting

- We consider $\eta = 0$.
- Compute $a_{\tau,m}$ by repeatedly applying Duhamel's formula

$$\mathrm{e}^{X+zY} = \mathrm{e}^{X} + z\int_0^1 \mathrm{d}t\, \mathrm{e}^{X(1-t)} Y \mathrm{e}^{t(X+zY)}$$

for $X = -H_{\tau,0}$ and $Y = -W_\tau$ in expansion of $A_\tau(z)$.
- We hence obtain

$$a_{\tau,m} = \frac{1}{\mathrm{Tr}\left(\mathrm{e}^{-H_{\tau,0}}\right)} \mathrm{Tr}\Bigg((-1)^m \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{m-1}} \mathrm{d}t_m$$

$$\mathrm{e}^{-(1-t_1)H_{\tau,0}}\, W_\tau\, \mathrm{e}^{-(t_1-t_2)H_{\tau,0}}\, W_\tau \cdots \mathrm{e}^{-(t_{m-1}-t_m)H_{\tau,0}}\, W_\tau\, \mathrm{e}^{-t_m H_{\tau,0}}\Bigg).$$

The quantum Wick theorem

- Rewrite $a_{\tau,m}$ using the *quantum Wick theorem*

$$\frac{1}{\mathrm{Tr}(\mathrm{e}^{-H_{\tau,0}})}\,\mathrm{Tr}\Big(\phi_\tau^*(x_1)\cdots\phi_\tau^*(x_k)\phi_\tau(y_1)\cdots\phi_\tau(y_k)\,\mathrm{e}^{-H_{\tau,0}}\Big)$$

$$= \sum_{\pi\in\mathcal{S}^k}\prod_{j=1}^{k}\frac{1}{\mathrm{Tr}(\mathrm{e}^{-H_{\tau,0}})}\,\mathrm{Tr}\Big(\phi_\tau^*(x_j)\phi_\tau(y_{\pi(j)})\,\mathrm{e}^{-H_{\tau,0}}\Big)\,.$$

- Factors are

$$\frac{1}{\mathrm{Tr}(\mathrm{e}^{-H_{\tau,0}})}\,\mathrm{Tr}\Big(\phi_\tau^*(x)\phi_\tau(y)\,\mathrm{e}^{-H_{\tau,0}}\Big) \;=\; G_\tau(x;y)\,,$$

where

$$G_\tau \;=\; \frac{1}{\tau\big(\mathrm{e}^{(-\Delta+\kappa)/\tau}-1\big)}$$

is the *quantum Green function*.
- Compare with the *classical Green function*

$$G \;=\; (-\Delta+\kappa)^{-1}\,.$$

The graph structure

- The pairing of ϕ_τ^*, ϕ_τ gives rise to a *graph structure*.
  $\rightarrow$ *Join vertices according to quantum Wick theorem*.
- $2m$ copies of ϕ_τ^*, $2m$ copies of ϕ_τ.
  $\rightarrow$ Total number of graphs is at most $(2m)! = \mathcal{O}(C^m m!^2)$.
- **Main work:** For fixed $t_1, \ldots, t_m$, each graph contributes $\mathcal{O}(C^m)$.
- Obtain gain of $\frac{1}{m!}$ from the time integral

$$\int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{m-1}} \mathrm{d}t_m = \frac{1}{m!}\,.$$

- Conclude that $|a_{\tau,m}| \leqslant C^m m!$.

The graph structure

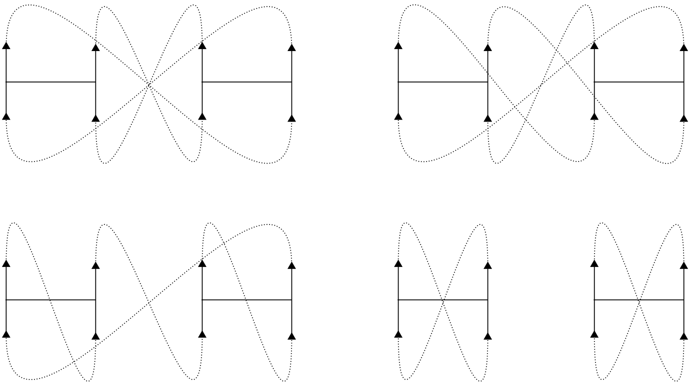

FIGURE. Some examples of the possible graphs when $m = 2$ in $2D$ and $3D$.
No two vertically adjacent vertices are joined due to Wick ordering.

The graph structure

From the *iterated integral structure*, it follows that we need to consider *time-evolved operators*

$$G_{\tau,t} := \frac{e^{-\frac{t}{\tau}(-\Delta+\kappa)}}{\tau\left(e^{(-\Delta+\kappa)/\tau} - 1\right)} \qquad (t \geqslant -1)\,.$$

$$S_{\tau,t} := e^{-\frac{t}{\tau}(-\Delta+\kappa)} \qquad (t \geqslant 0)\,,$$

In particular, $G_{\tau,0} = G_\tau$ and $S_{\tau,0} = \mathrm{I}$.

Lemma

For all $t > -1$ *and* $x, y \in \Lambda$ *we have* $G_{\tau,t}(x;y) \geqslant 0$ *.*
Moreover, for all $t \geqslant 0$ *and* $x, y \in \Lambda$ *we have* $S_{\tau,t}(x;y) \geqslant 0$ *.*

The graph structure

- **Difficulty:** The Hilbert-Schmidt norm $\|G_{\tau,t}\|_{HS}$ is *not bounded uniformly in $t > -1$*.
- We can only bound the operator norm uniformly in t.
- **Solution:** Use positivity.
- **Example:** Consider

$$\int_{\mathbb{T}^d} \mathrm{d}x_1 \int_{\mathbb{T}^d} \mathrm{d}x_2 \, w(x_1 - y_1) \, G_{\tau,t}(x_1; x_2) \, w(x_2 - y_2) \, G_{\tau,-t}(x_2; x_1) \, .$$

In absolute value, this is

$$\leqslant \|w\|^2_{L^\infty(\mathbb{T}^d)} \int_{\mathbb{T}^d} \mathrm{d}x_1 \int_{\mathbb{T}^d} \mathrm{d}x_2 \, G_{\tau,t}(x_1; x_2) \, G_{\tau,-t}(x_2; x_1) \, ,$$

which is

$$= \|w\|^2_{L^\infty(\mathbb{T}^d)} \int_{\mathbb{T}^d} \mathrm{d}x_1 \int_{\mathbb{T}^d} \mathrm{d}x_2 \, G_{\tau,0}(x_1; x_2) \, G_{\tau,0}(x_2; x_1)$$
$$= \|w\|^2_{L^\infty(\mathbb{T}^d)} \, \|G_{\tau,0}\|^2_{HS} \, .$$

$\rightarrow$ *Bounded uniformly in $t \geqslant -1, \tau \geqslant 1$.*

Time-dependent correlations

- Let $(\Gamma, H, \{\cdot,\cdot\})$ be a Hamiltonian system.
- $\mu(\mathrm{d}\phi) := \frac{1}{Z}\mathrm{e}^{-H(\phi)}\,\mathrm{d}\phi$, the associated Gibbs measure.
- $S_t :=$ flow map of H.
- Given $m \in \mathbb{N}$, observables $X^1, \dots, X^m \in C^\infty(\Gamma)$, and times $t_1, \dots, t_m \in \mathbb{R}$, define the m***-particle time-dependent correlation function***

$$\mathcal{Q}_\mu(X^1, \dots, X^m; t_1, \dots, t_m) := \int X^1(S_{t_1}\phi) \cdots X^m(S_{t_m}\phi)\,\mathrm{d}\mu\,.$$

- **Goal:** Obtain a derivation of $\mathcal{Q}_\mu$ from many-body quantum expectation values in the setting where S_t *is the flow of the (cubic) NLS on* $\mathbb{T}^1$. S_t is globally defined on $\Gamma := L^2(\mathbb{T}^1)$ (Bourgain, 1993).

Time evolution of observables

- Let $\xi \in \mathcal{L}(\mathfrak{H}^{(p)})$ be given. Define the lift of ξ to an operator on $\mathcal{F}$

$$\Theta_\tau(\xi) := \int \mathrm{d}x_1 \dots \mathrm{d}x_p \, \mathrm{d}y_1 \dots \mathrm{d}y_p \, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \\ \phi_\tau^*(x_1) \cdots \phi_\tau^*(x_p) \, \phi_\tau(y_1) \cdots \phi_\tau(y_p) \, .$$

- The *time-evolved* operator $\Psi_\tau^t \Theta_\tau(\xi) := \mathrm{e}^{\mathrm{i}t\tau H_\tau} \, \Theta_\tau(\xi) \, \mathrm{e}^{-\mathrm{i}t\tau H_\tau}$.
- The *random variable*

$$\Theta(\xi) := \int \mathrm{d}x_1 \dots \mathrm{d}x_p \, \mathrm{d}y_1 \dots \mathrm{d}y_p \, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \\ \bar{\phi}(x_1) \cdots \bar{\phi}(x_p) \, \phi(y_1) \cdots \phi(y_p) \, .$$

- The *time-evolved* random variable

$$\Psi^t \Theta(\xi) := \int \mathrm{d}x_1 \dots \mathrm{d}x_p \, \mathrm{d}y_1 \dots \mathrm{d}y_p \, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \\ \overline{S_t \phi}(x_1) \cdots \overline{S_t \phi}(x_p) \, S_t \phi(y_1) \cdots S_t \phi(y_p) \, .$$

Microscopic derivation of time-dependent correlations

Theorem 2: Fröhlich, Knowles, Schlein, S. (preprint 2017).

Let $w \in L^\infty(\mathbb{T}^1)$ be pointwise nonnegative.
Given $m \in \mathbb{N}$, $\xi^j \in \mathcal{L}(\mathfrak{H}^{(p_j)})$ and times t_j, we have

$$\rho_\tau\big(\Psi_\tau^{t_1}\Theta_\tau(\xi^1) \cdots \Psi_\tau^{t_m}\Theta_\tau(\xi^m)\big) \to \rho\big(\Psi^{t_1}\Theta(\xi^1) \cdots \Psi^{t_m}\Theta(\xi^m)\big) \quad \text{as} \quad \tau \to \infty\,,$$

Theorem 1 in $1D$ corresponds to Theorem 2 with $m = 1$ and $t_1 = 0$.
Use

$$\rho_\tau(\Theta_\tau(\xi)) - \rho(\Theta(\xi)) \;=\; \mathrm{Tr}\,\big((\gamma_{\tau,p} - \gamma_p)\xi\big)$$

for $\xi \in \mathcal{L}(\mathfrak{H}^{(p)})$ and argue by duality.

Idea of proof: Truncation in particle number

- Use an approximation argument and reduce to showing that

$$\rho_\tau\big(\Psi_\tau^{t_1}\Theta_\tau(\xi^1)\,\cdots\,\Psi_\tau^{t_m}\Theta_\tau(\xi^m)F(\mathcal{N}_\tau)\big) \to \rho\big(\Psi^{t_1}\Theta(\xi^1)\,\cdots\,\Psi^{t_m}\Theta(\xi^m)F(\mathcal{N})\big)$$

  for appropriately chosen $F \in C_c^\infty(\mathbb{R})$.
- The general claim can be shown to follow from

$$\rho_\tau\big(\Theta_\tau(\xi)F(\mathcal{N}_\tau)\big) \to \rho\big(\Theta(\xi)F(\mathcal{N})\big)\,.$$

- Presence of cut-off F *does not allow direct application of Wick theorem*.
- Expand $F(\mathcal{N}_\tau)$ and $F(\mathcal{N})$.

The local problem

- A first attempt does not immediately yield the result of Theorem 2 for the local (cubic) NLS

$$\mathrm{i}\partial_t \phi_t(x) + \Delta \phi_t(x) \;=\; |\phi_t(x)|^2\, \phi_t(x)\,,$$

i.e. for $w = \delta$.

- Instead, first prove Theorem 2 for the flow of the nonlocal (Hartree) equation

$$\mathrm{i}\partial_t \phi_t^\varepsilon(x) + \Delta \phi_t^\varepsilon(x) \;=\; \int \mathrm{d}y\, w^\varepsilon(x-y)\, |\phi_t^\varepsilon(y)|^2\, \phi_t^\varepsilon(x)\,,$$

with $w^\varepsilon \in L^\infty(\mathbb{T}^1)$, $w^\varepsilon \rightharpoonup \delta$ and with the same initial data.

- **Stability:** For sufficiently regular initial data

$$\|\phi_t^\varepsilon - \phi_t\|_{L^2(\mathbb{T}^1)} \to 0 \quad \text{as} \quad \varepsilon \to 0\,.$$

- For proof of stability, use ***dispersion of NLS***. Work in $X^{s,b}$ ***spaces***

$$\|u\|_{X^{s,b}} \;:=\; \|\mathrm{e}^{-\mathrm{i}t\Delta} u\|_{H_t^b H_x^s}\,.$$

Thank you for your attention!