L'Aquila - November 28, 2018

Writhe and twist helicity of quantum vortex systems

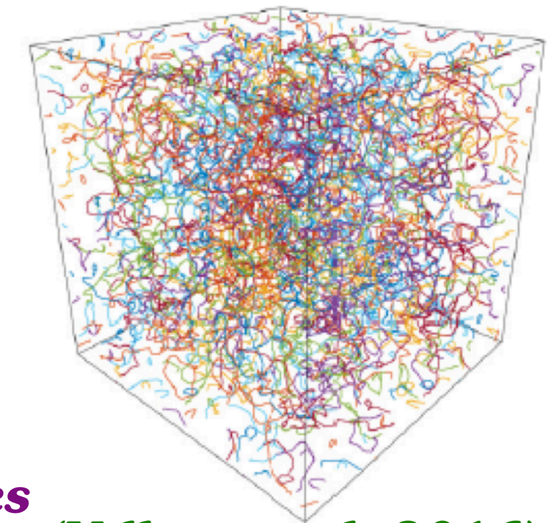
RENZO L. RICCA

Department of Mathematics & Applications, U. Milano-Bicocca

renzo.ricca@unimib.it

Motivations

- Relationships between morphological and energetic aspects of defects (quantum vortices)
- Topological properties and stability of defects
- Reconnection mechanisms and energy transfer
- Production of new defects from existing ones
- Topological cascade processes of defects
- Entanglement and structural complexity measures
- Quantum versus classical turbulence



(Villois et al. 2016)

From condensates to quantum fluids

- ***Bose-Einstein condensate:***
  - *state of matter in which a diluted gas of bosons (such as photons and ^{4}He atoms) at extremely low density is cooled at at temperatures of nano-kelvins and coalesces into a quantum mechanical state that can be described by a single wave function on a near-macroscopic scale;*
  - *1924: first theoretical prediction by Bose & Einstein;*
  - *1938: London's proposal for superfluidity;*
  - *1995: first experimental production of a BEC of rubidium atoms by Cornell, Wieman & Ketterle (2001 Physics Nobel Prize).*

- ***Gross-Pitaevskii equation (GPE):***
  - *1941: mean-field theory from first-order quantization;*
  - *1961-1963: Pitaevskii & Gross' derivation in terms of non-linear Schrödinger equation (NLSE).*

- ***Hydrodynamical interpretation:***
  - *1926: Madelung transformation of (NL)SE into hydrodynamic form;*
  - *1952: Bohm's theory of quantum trajectories;*
  - *1980s: implementation of numerical method for quantum fluids;*
  - *2010s: study of topological stability and structural complexity issues.*

Quantum hydrodynamics under the Gross-Pitaevskii equation (GPE)

- ***Gross-Pitaevskii equation:***

$$\frac{\partial \psi}{\partial t} = \frac{\mathrm{i}}{2}\nabla^2\psi + \frac{\mathrm{i}}{2}(1 - |\psi|^2)\psi \quad \textit{with} \quad |\psi|^2 \to 1 \ \textit{as} \ \mathbf{X} \to \infty \ .$$

- ***Madelung transformation:*** $\psi = \sqrt{\rho}\ e^{i\theta}$ $\quad \begin{cases} \rho = |\psi|^2 \\ \mathbf{u} = \nabla\theta \end{cases}$

- ***Hydrodynamical interpretation of GPE:***

GPE:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho u_k)}{\partial x_k} = 0, & \textit{(cotinuity eq.)} \\ \rho\left(\dfrac{\partial u_k}{\partial t} + u_l \dfrac{\partial u_k}{\partial u_l}\right) = -\dfrac{\partial p}{\partial x_k} + \dfrac{\partial \tau_{kl}}{\partial x_l}\ , & \textit{(momentum eq.)} \end{cases}$$

- ***Conserved hamiltonian (to leading order):***

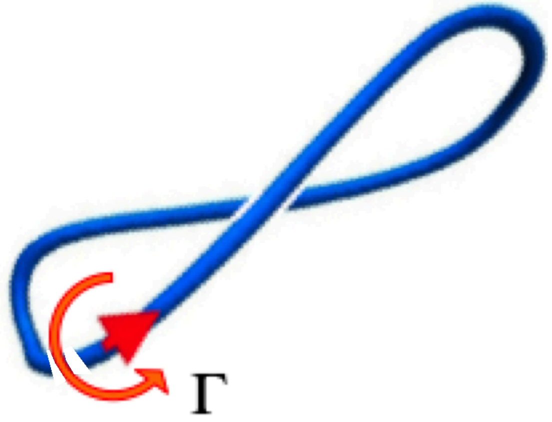
$$H = K + I = \text{constant} \quad \begin{cases} K = \dfrac{1}{2}\displaystyle\int \nabla\psi\cdot\nabla\psi^* \mathrm{d}^3\mathbf{X} \\ I = \dfrac{1}{4}\displaystyle\int \left(1 - |\psi|^2\right)^2 \mathrm{d}^3\mathbf{X} \end{cases}$$

Helicity and linking numbers

- ***Helicity*** $H(t)$ ***:***

$$H(t) = \int_{V(\omega)} \mathbf{u}\cdot\boldsymbol{\omega}\, \mathrm{d}^3\mathbf{X}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ ***, with*** $\nabla\cdot\mathbf{u} = 0$ ***in*** $\mathbb{R}^3$***.***



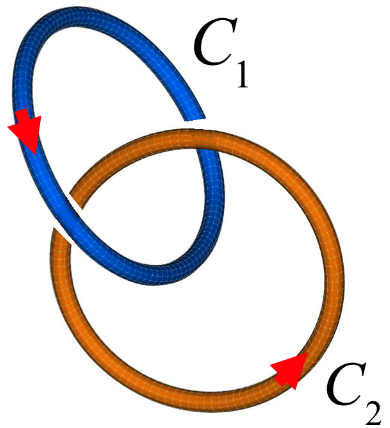
- ***Theorem (Moreau 1961). Under ideal conditions helicity is a quantity conserved during evolution, that is***

$$\frac{dH(t)}{dt} = 0 \quad \to \quad H = \text{constant} .$$

- ***Theorem (Moffatt 1969; Moffatt & Ricca 1992). Let*** $\mathcal{L}_{\mathrm{n}}$ ***be a disjoint union of vortex tubes in an ideal fluid. Then, we have***

$$H(\mathcal{L}_{\mathrm{n}}) = \int_{V(\omega)} \mathbf{u}\cdot\boldsymbol{\omega}\, \mathrm{d}^3\mathbf{X} = \sum_{\mathrm{i}\neq\mathrm{j}} Lk_{\mathrm{ij}}\Gamma_{\mathrm{i}}\Gamma_{\mathrm{j}} + \sum_{\mathrm{i}} Lk_{\mathrm{i}}\Gamma_{\mathrm{i}}^2$$

$$= \sum_{\mathrm{i}\neq\mathrm{j}} Lk_{\mathrm{ij}}\Gamma_{\mathrm{i}}\Gamma_{\mathrm{j}} + \sum_{\mathrm{i}} \left(Wr + Tw\right)\Gamma_{\mathrm{i}}^2 .$$



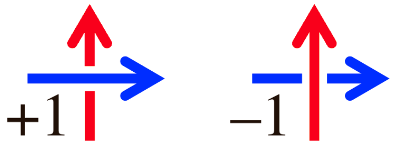
Gauss linking number

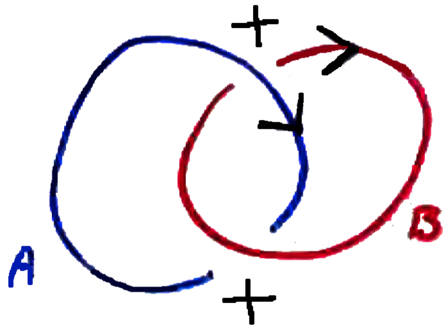
- ***Gauss linking number:*** $Lk_{12} = Lk(C_1, C_2) = \dfrac{1}{4\pi}\oint_{C_1}\oint_{C_2}\dfrac{(\mathbf{x}_1-\mathbf{x}_2)\cdot d\mathbf{x}_1\times d\mathbf{x}_2}{|\mathbf{x}_1-\mathbf{x}_2|^3}$

  (Gauss 1833)

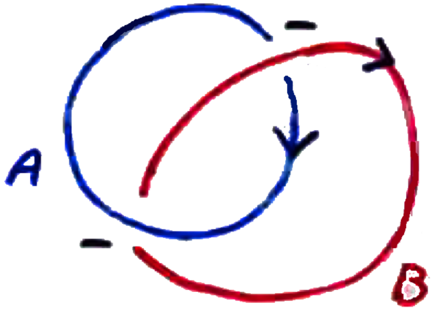
- ***Algebraic definition:*** $Lk = \dfrac{1}{2}\sum_r \varepsilon_r$

$\varepsilon_r = \pm 1$

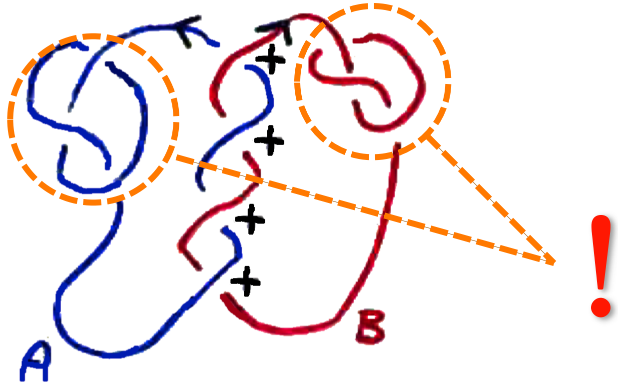




$Lk(A,B) = \dfrac{+2}{2} = +1$



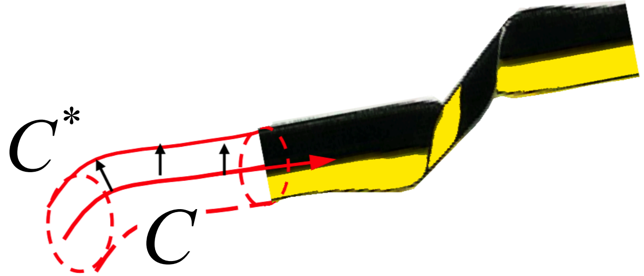
$Lk(A,B) = \dfrac{-2}{2} = -1$



$Lk(A,B) = \dfrac{+4}{2} = +2$

Călugăreanu-White invariant

From the Gauss linking number $Lk_{12} = Lk(C_1, C_2)$ ***take*** $C_1 = C$ ***and*** $C_2 = C^*$ ***; define the ribbon*** $R = R(C, C^*)$ ***by***



$$R = R(C, C^*) \quad \begin{cases} C: \ \mathbf{x} = \mathbf{x}(s) \\ C^*: \ \mathbf{x}^* = \mathbf{x}(s) + \varepsilon\,\hat{\mathbf{N}}(s) \end{cases}$$

and take $\lim\limits_{\varepsilon \to 0} Lk(C, C^*) = Lk(R)$.

- ***Călugăreanu-White invariant:***

$$Lk(R) = Wr(C) + Tw(R)$$

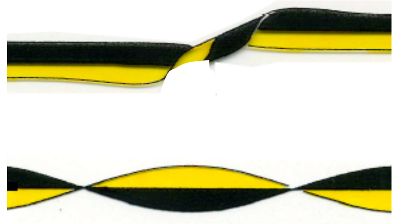
(Călugăreanu 1961; White 1969)

writhing number:

$$Wr(C) = \frac{1}{4\pi} \oint_C \oint_C \frac{(\mathbf{x} - \mathbf{x}^*) \cdot \mathrm{d}\mathbf{x} \times \mathrm{d}\mathbf{x}^*}{|\mathbf{x} - \mathbf{x}|^3}$$

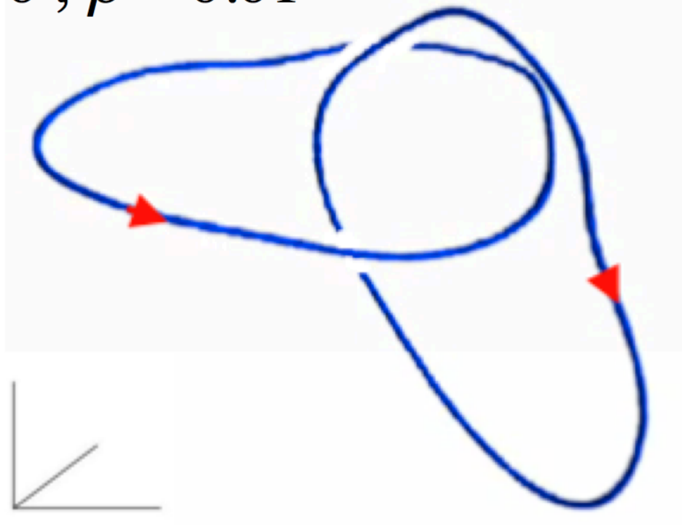
total twist number:

$$Tw(R) = \frac{1}{2\pi} \oint_R \left(\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} \times \hat{\mathbf{N}} \right) \cdot \hat{\mathbf{t}} \ \mathrm{d}s$$

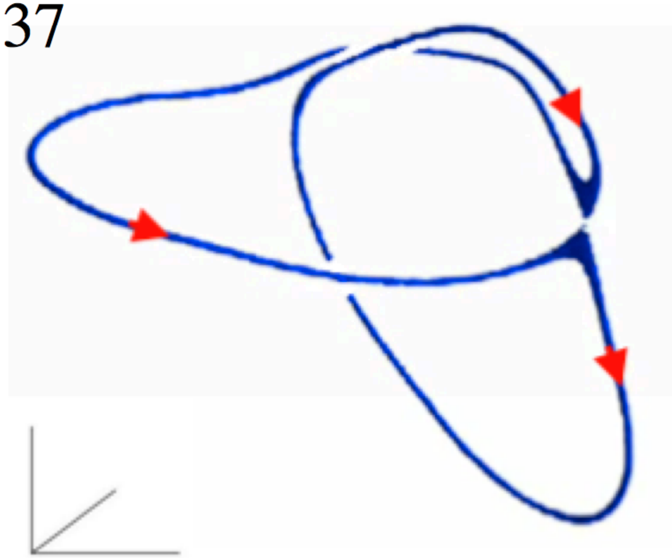


Evolution of a Hopf link of quantum vortices ($\Gamma = 1$) under GPE

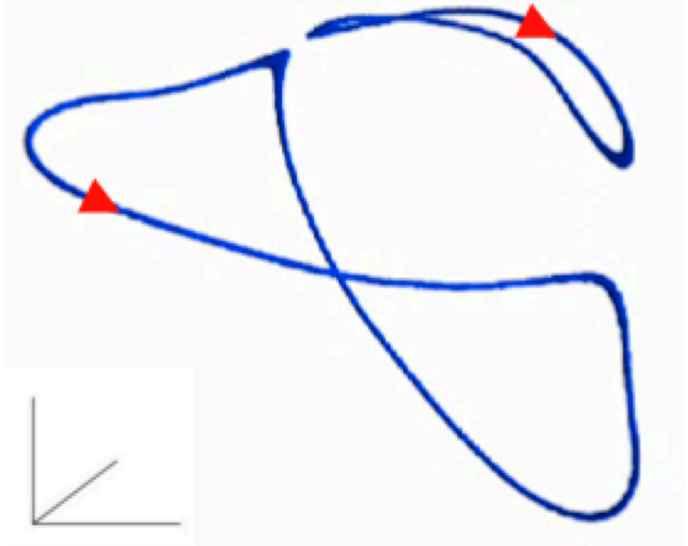
$t = 30$, $\rho = 0.01$



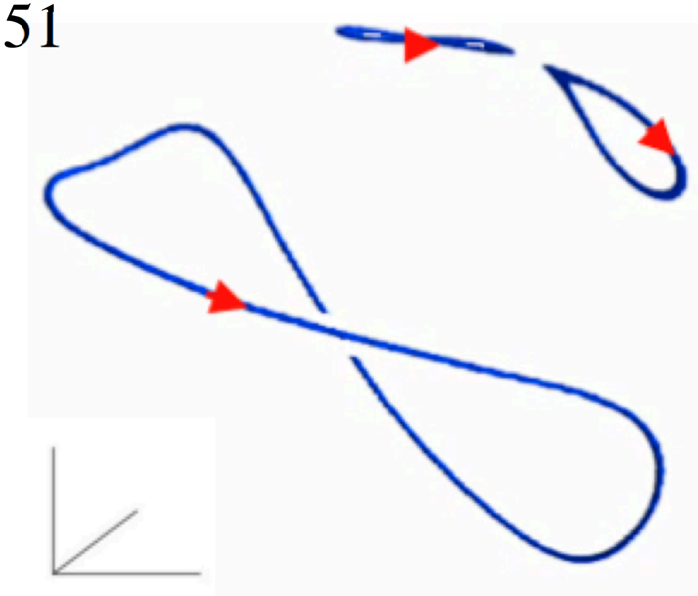
$t = 37$



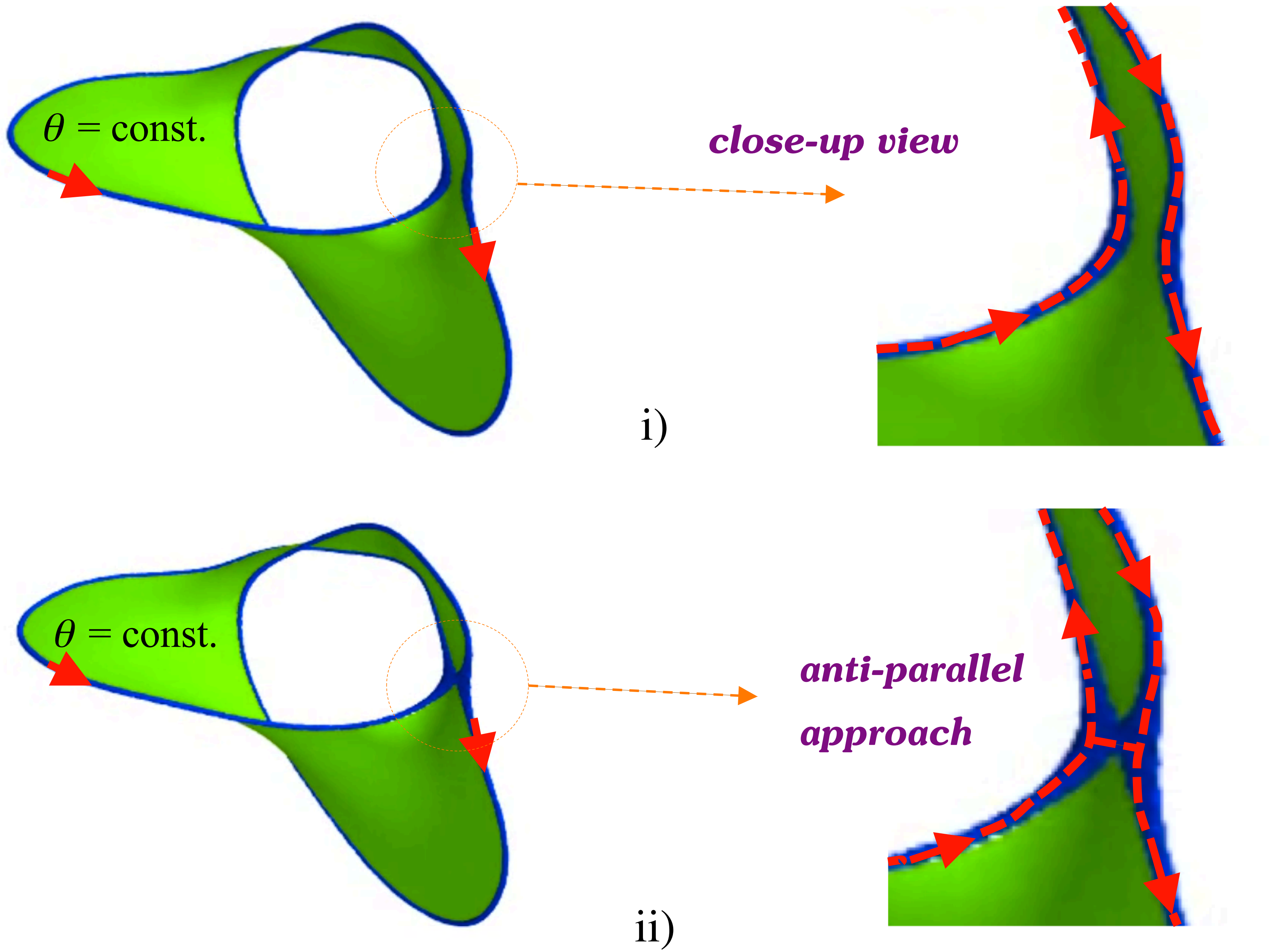
$t = 41.5$



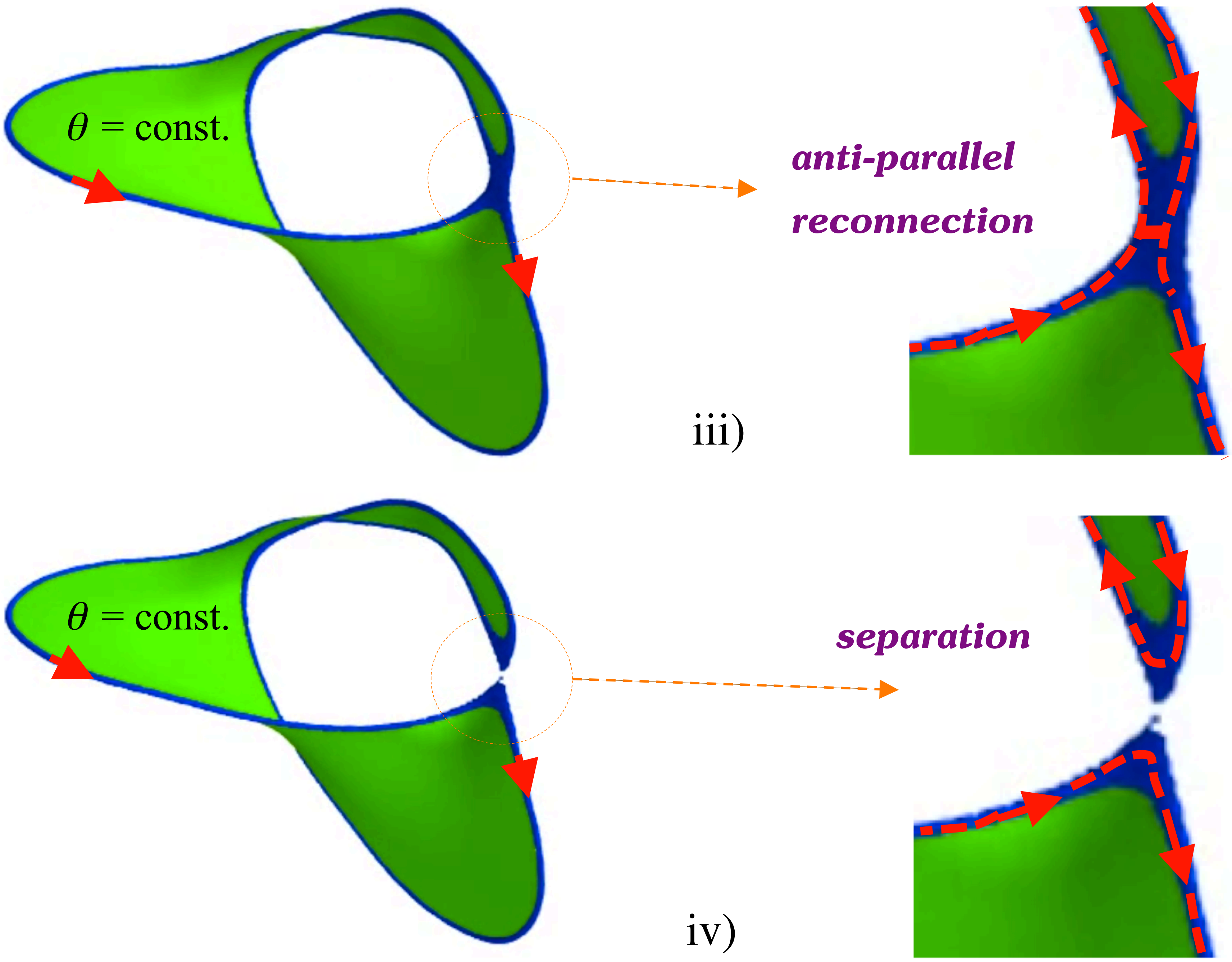
$t = 51$



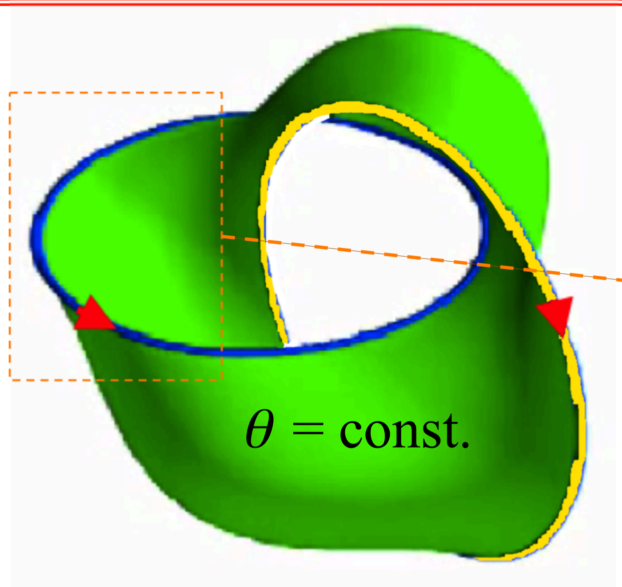
Reconnection process in terms of the iso-phase surface

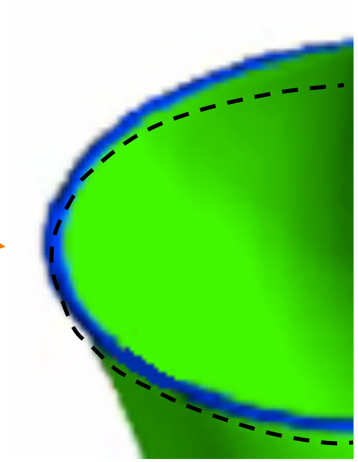


Reconnection process in terms of the iso-phase surface (cont.)

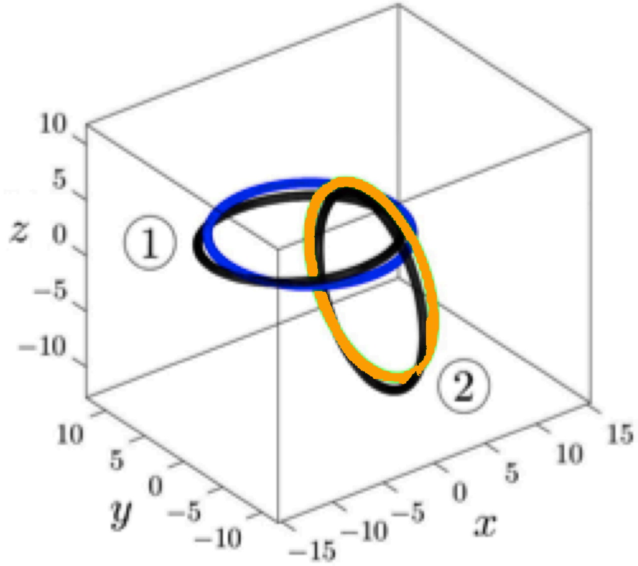


Twist analysis by ribbon construction on isophase surface

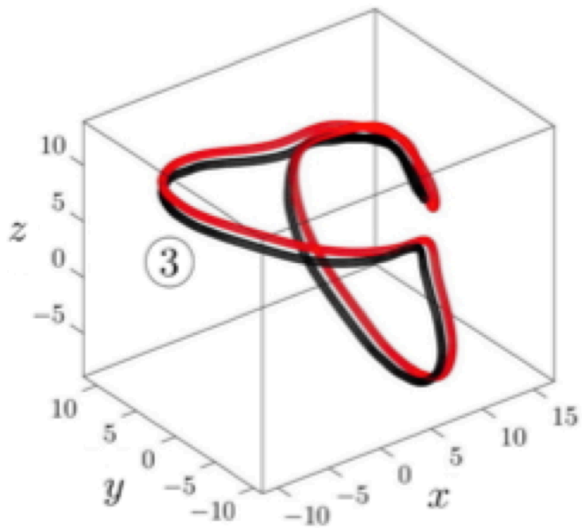




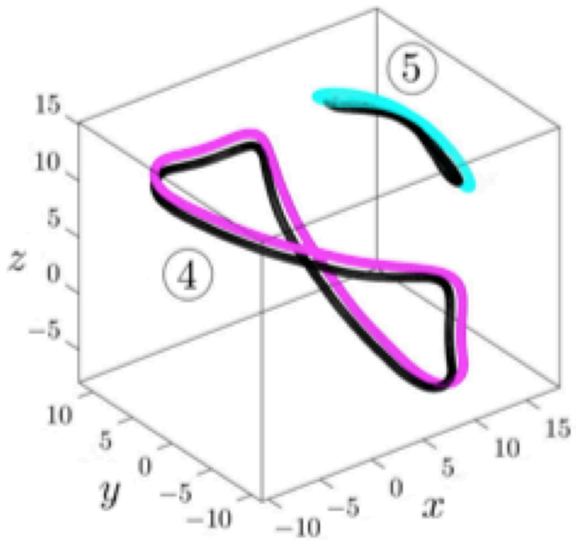
ribbon construction



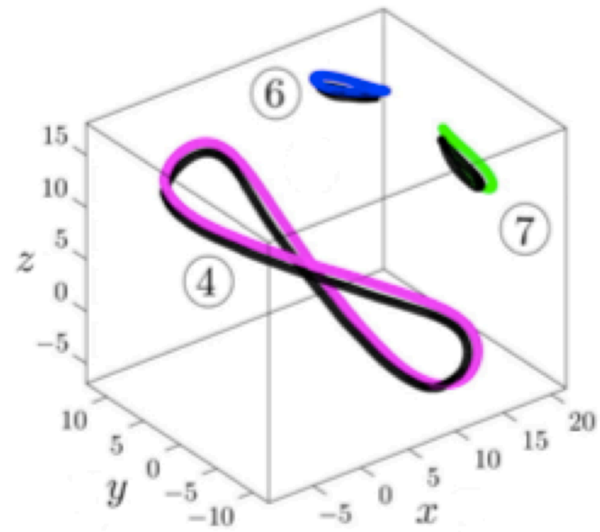
$t = 0$



$t = 38$

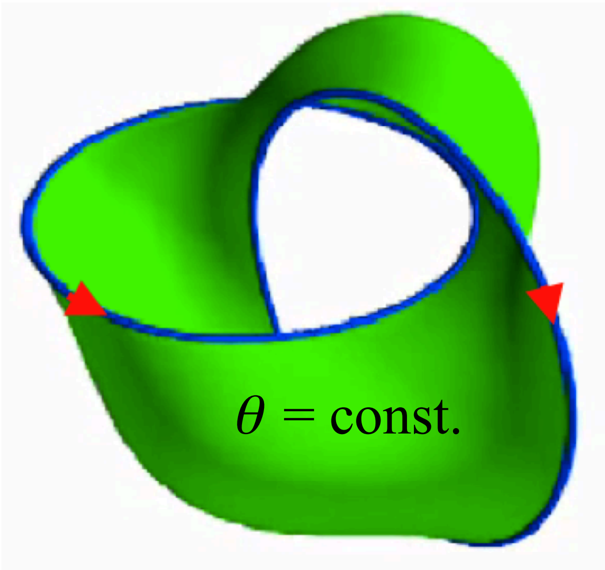


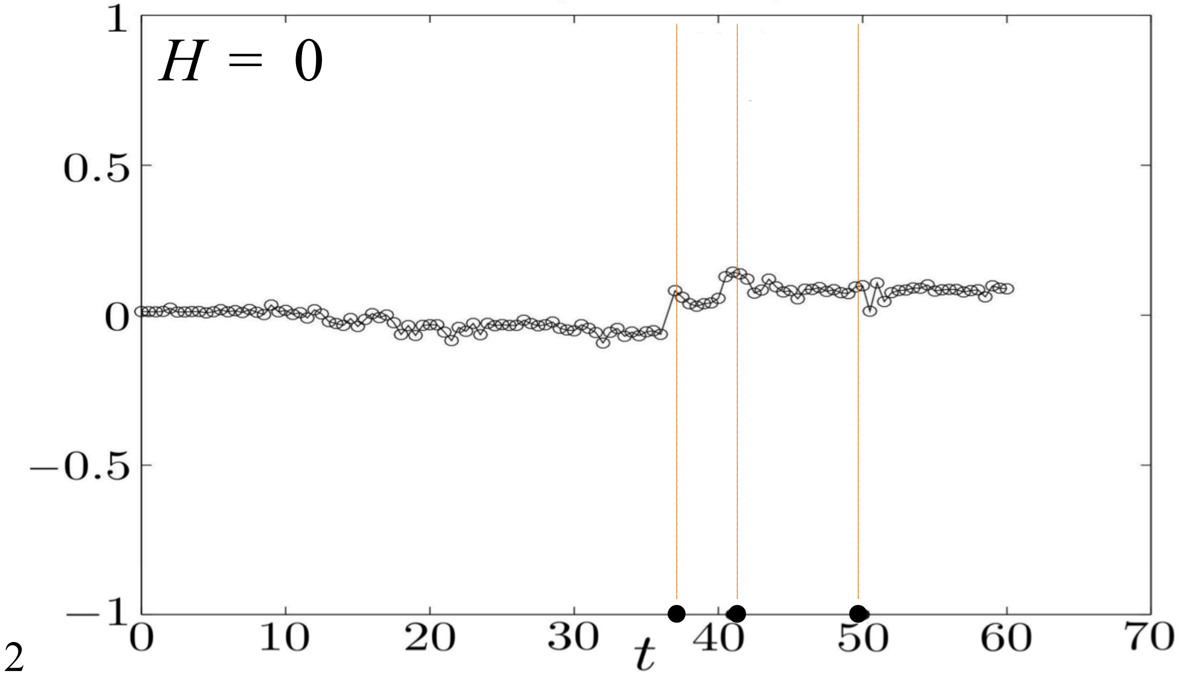
$t = 47$



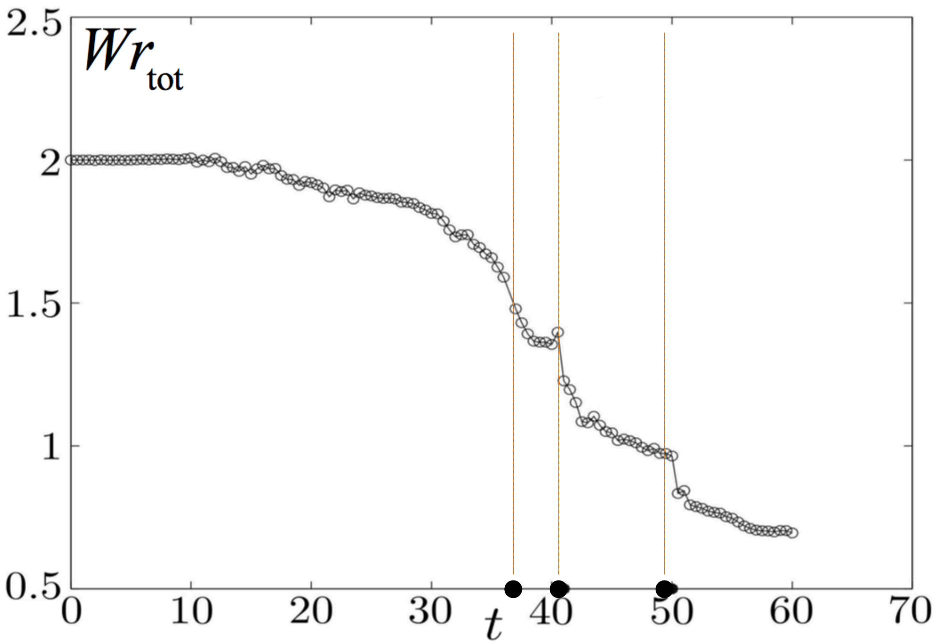
$t = 57$

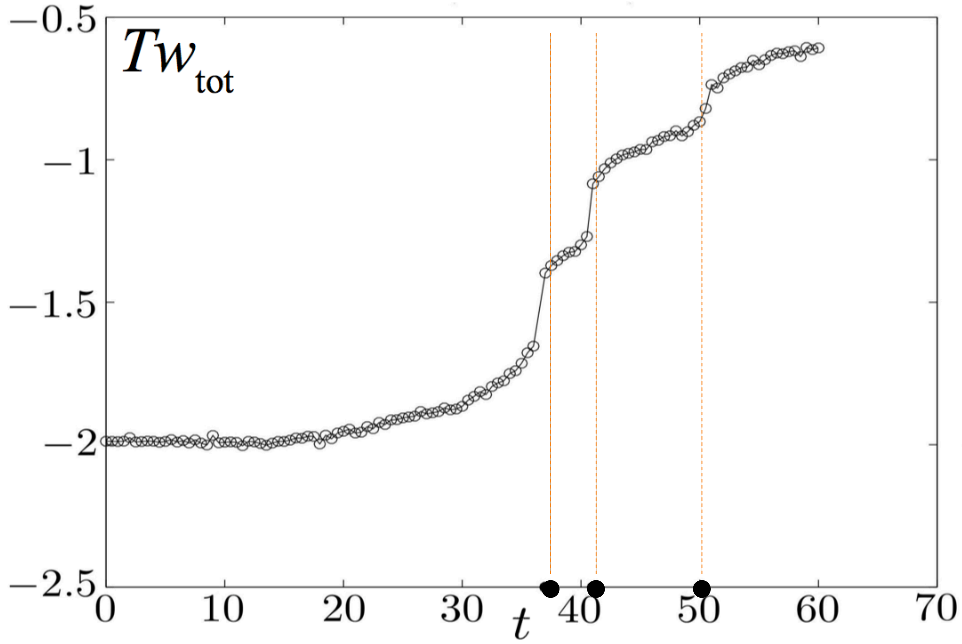
Writhe and twist contributions to helicity (Zuccher & Ricca PRE 2017)



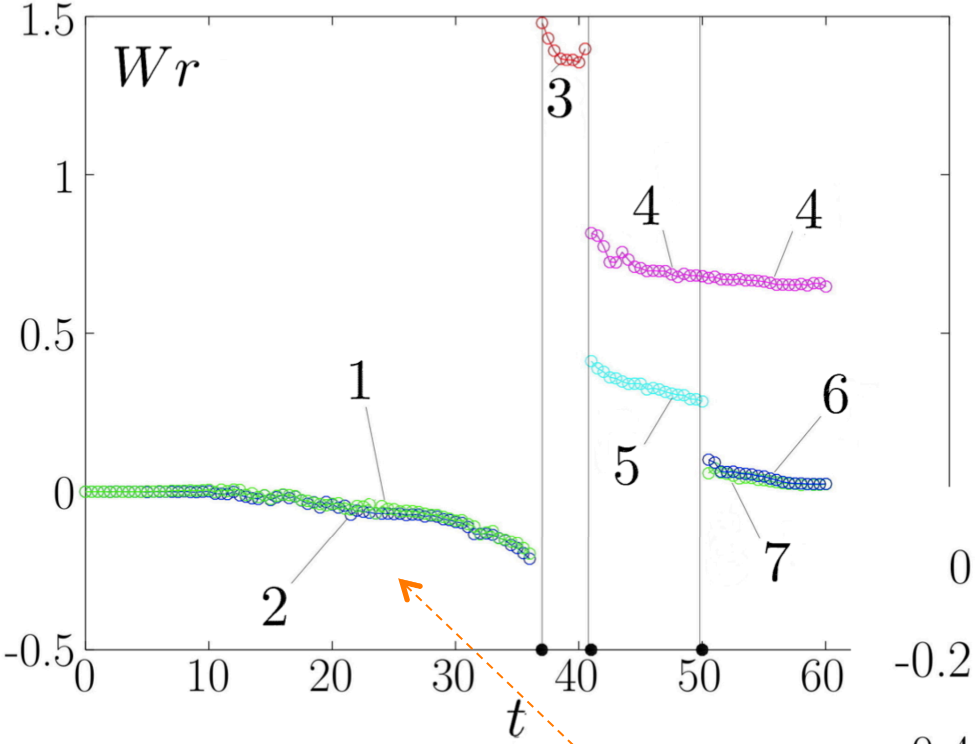


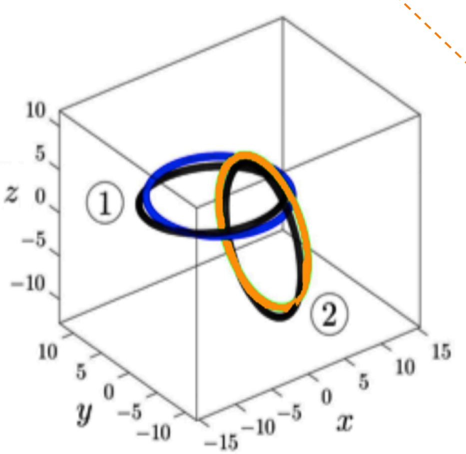
$$Wr_{\mathrm{tot}} = Wr_1 + Wr_2 + 2Lk_{12}$$

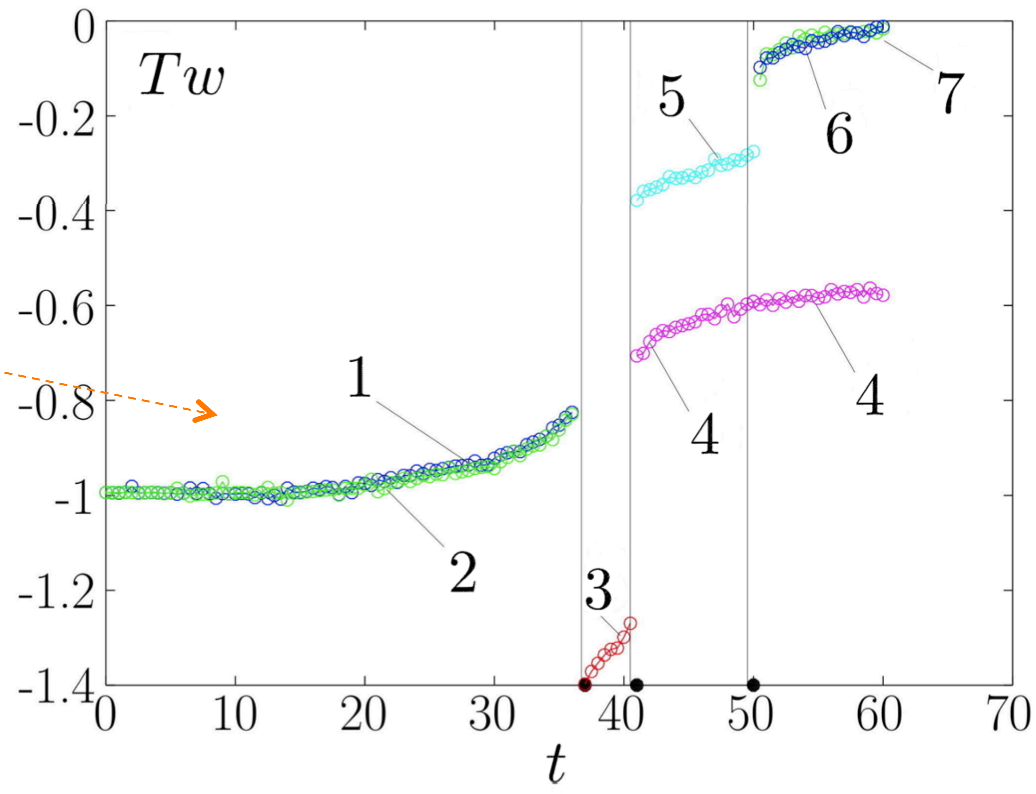




Individual writhe and twist contributions

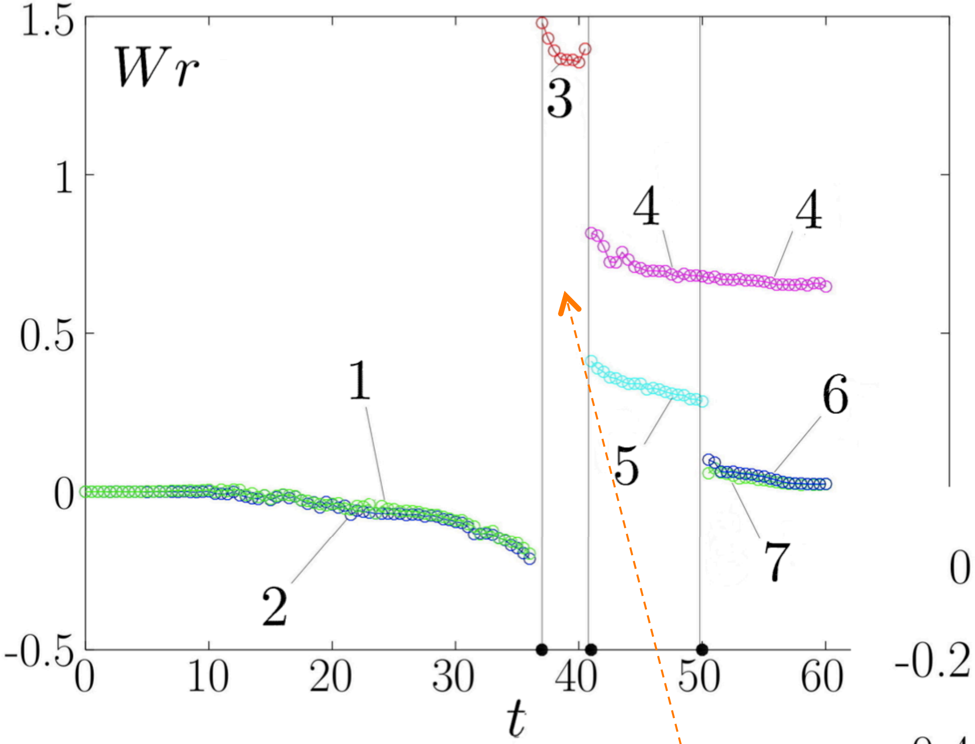


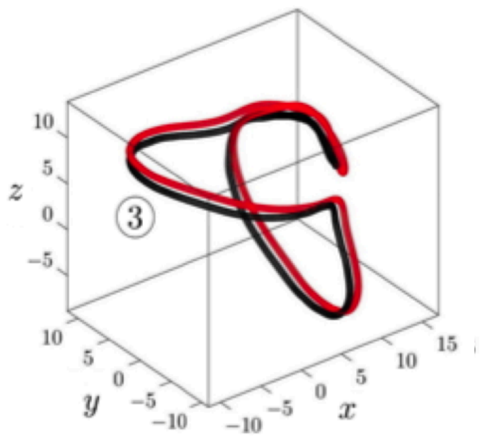


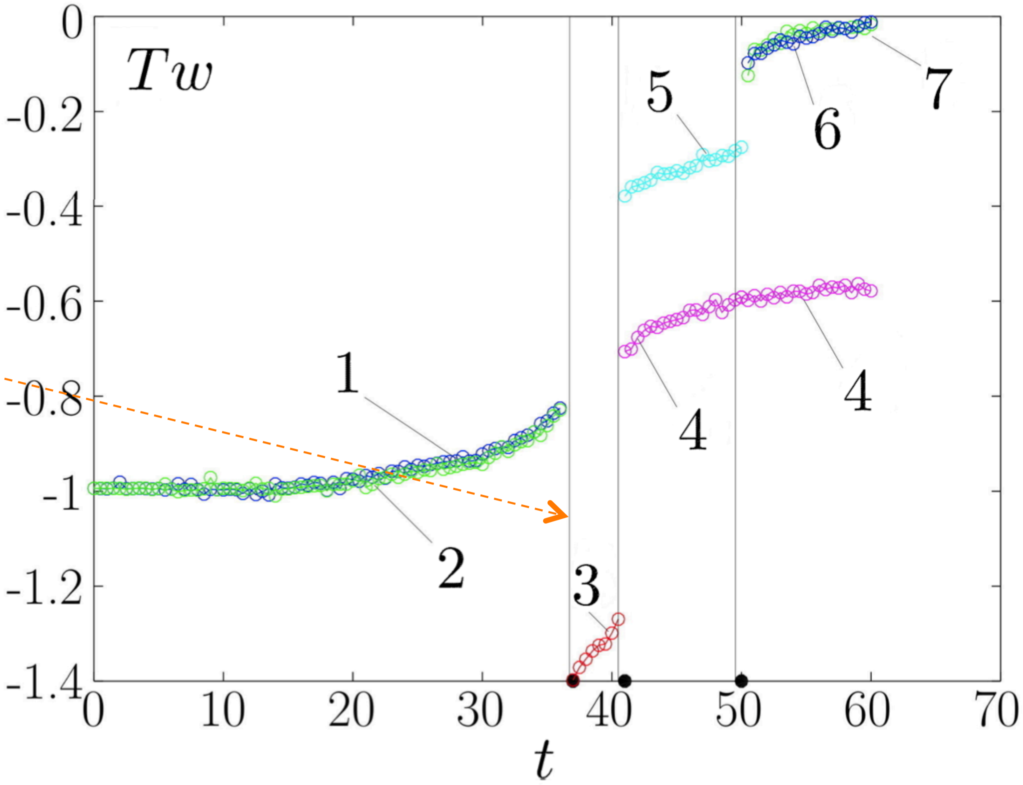


(Zuccher & Ricca PRE 2017)

Individual writhe and twist contributions

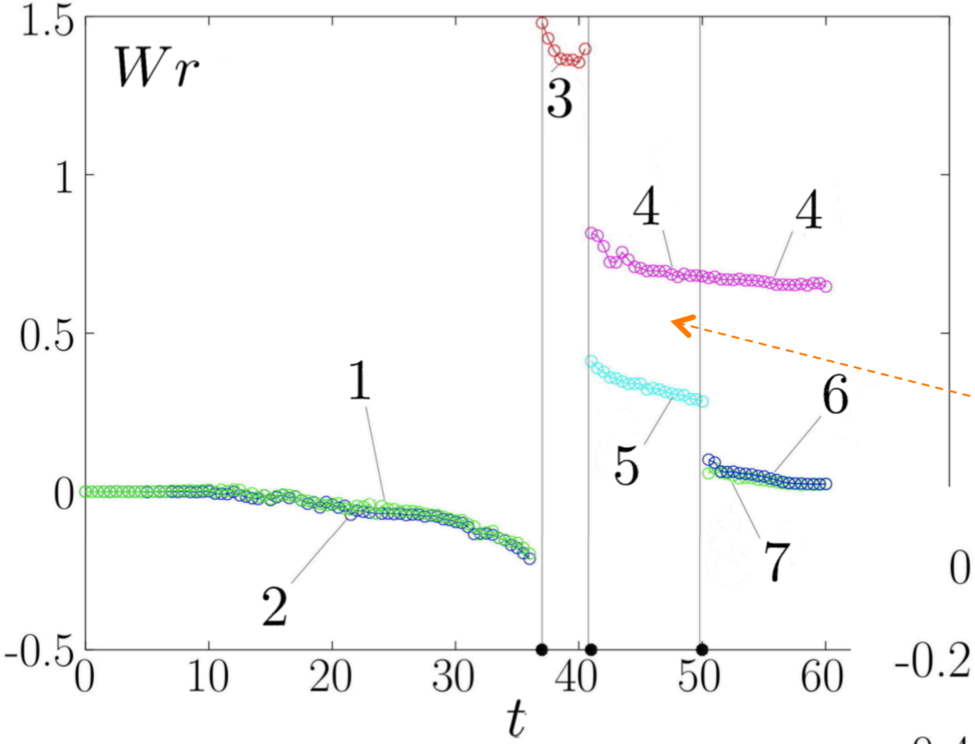


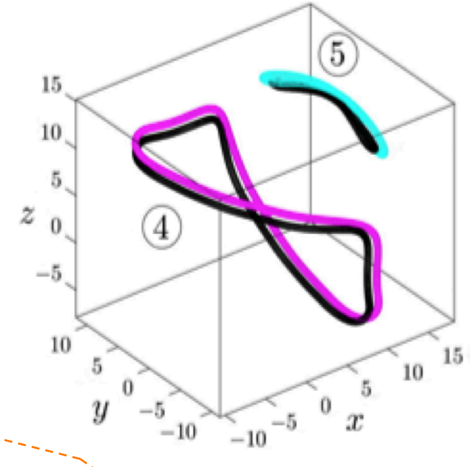


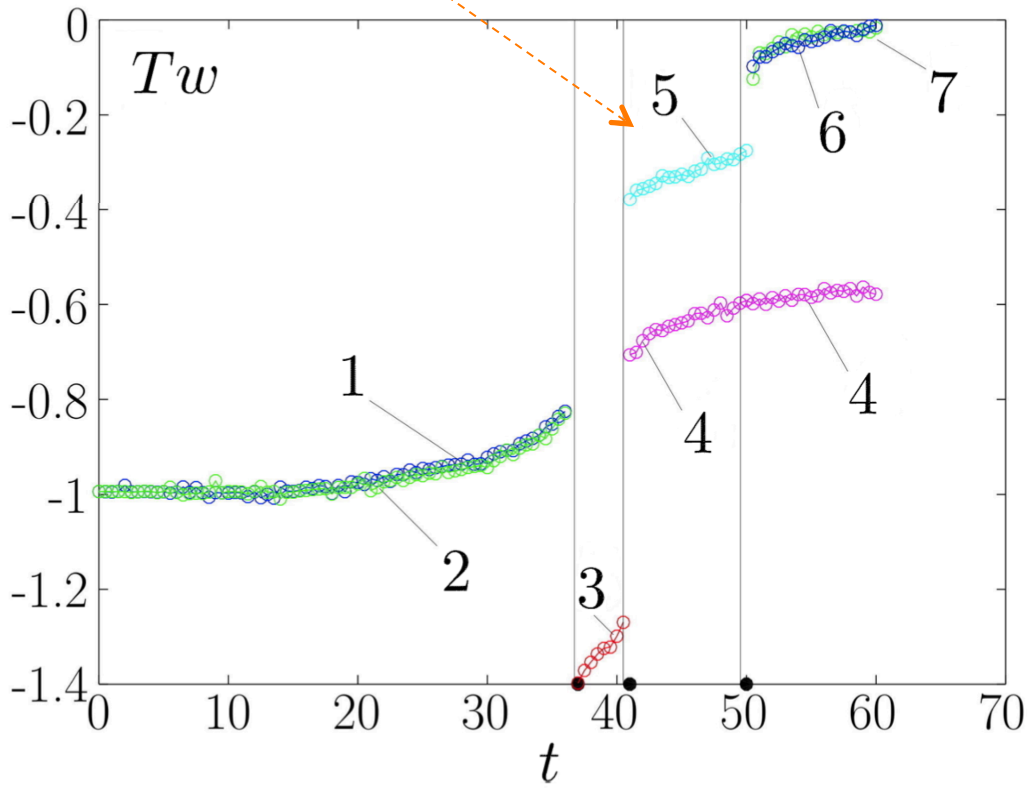


(Zuccher & Ricca PRE 2017)

Individual writhe and twist contributions

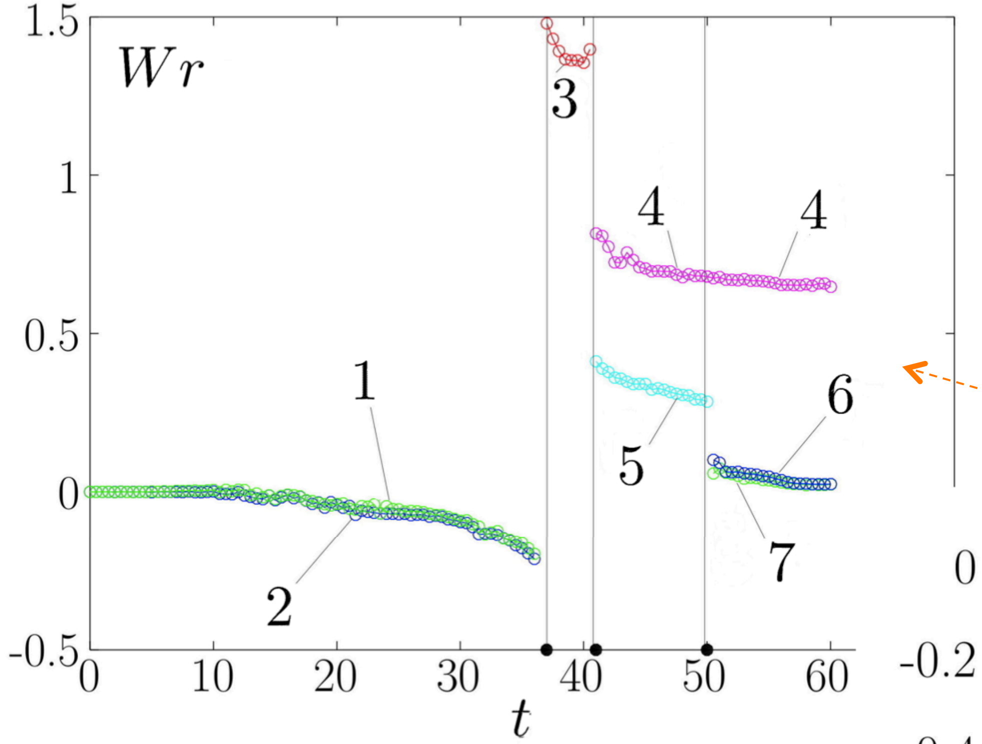


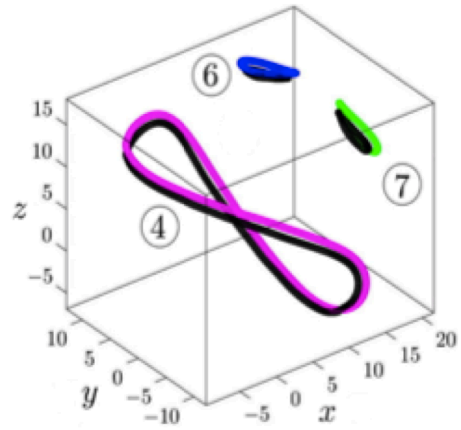


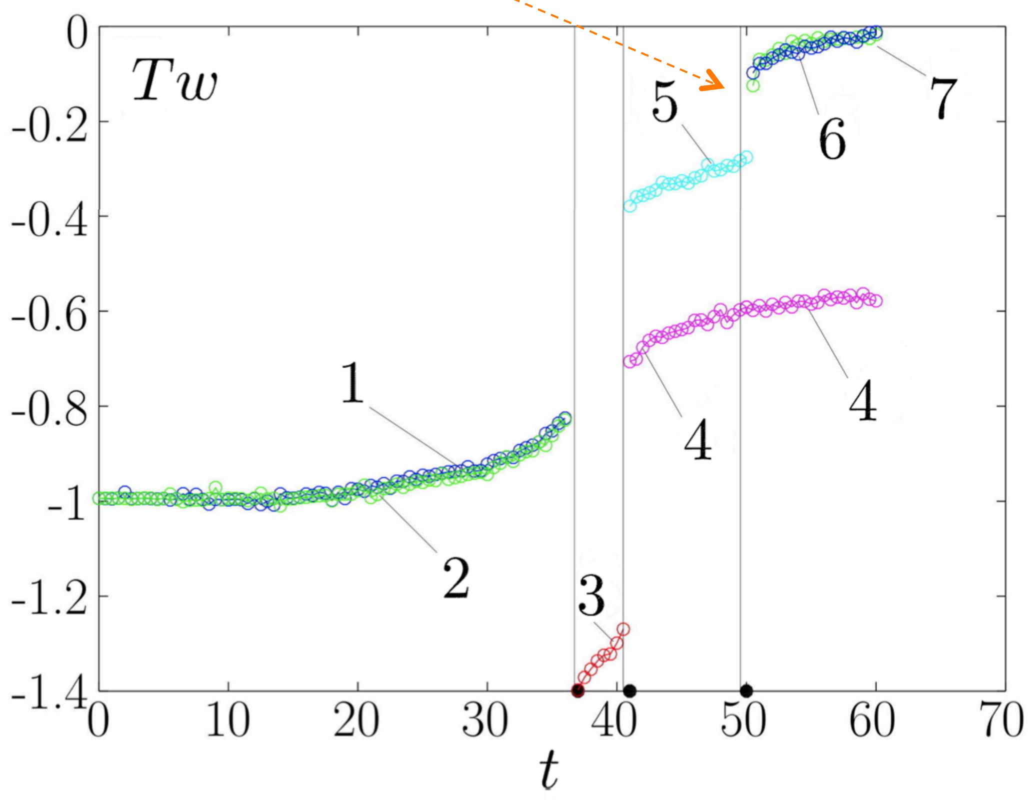


(Zuccher & Ricca PRE 2017)

Individual writhe and twist contributions



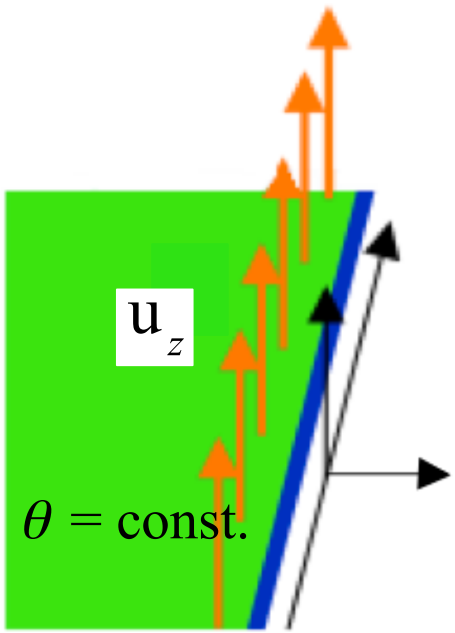




(Zuccher & Ricca PRE 2017)

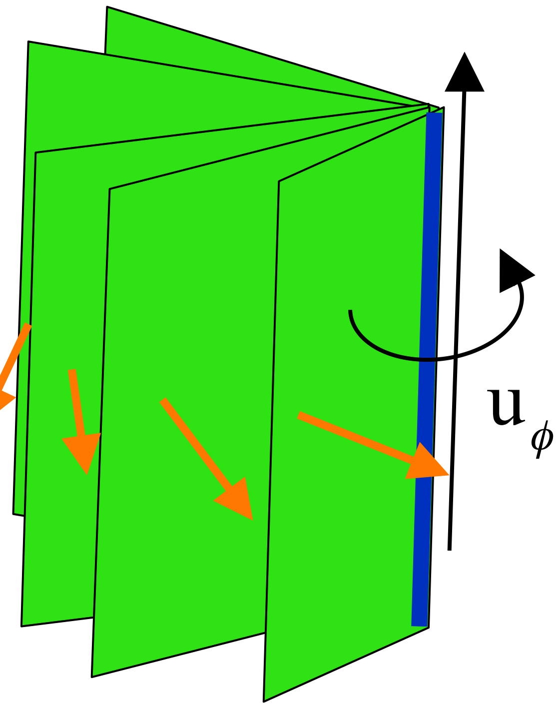
Physical interpretation of twisted iso-phase surface

Since $\mathbf{u} = \nabla\theta$ ***, we have:***



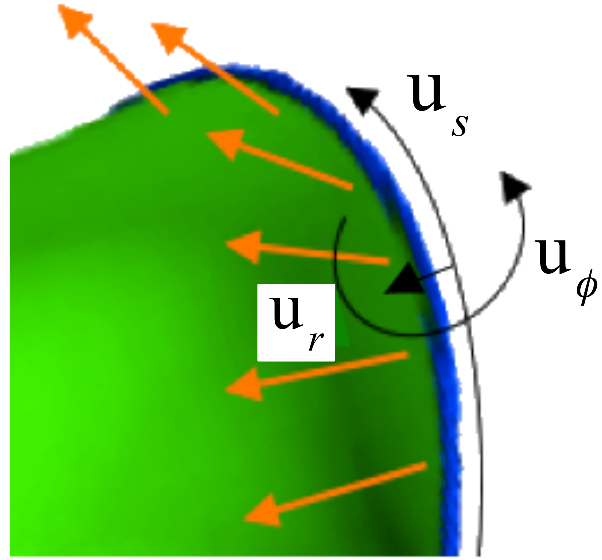
$$\mathbf{u} = u_z \hat{\mathbf{e}}_z$$

straight line defect in the plane



$$\mathbf{u} = u_\phi \hat{\mathbf{e}}_\phi$$

"open book" decomposition in space



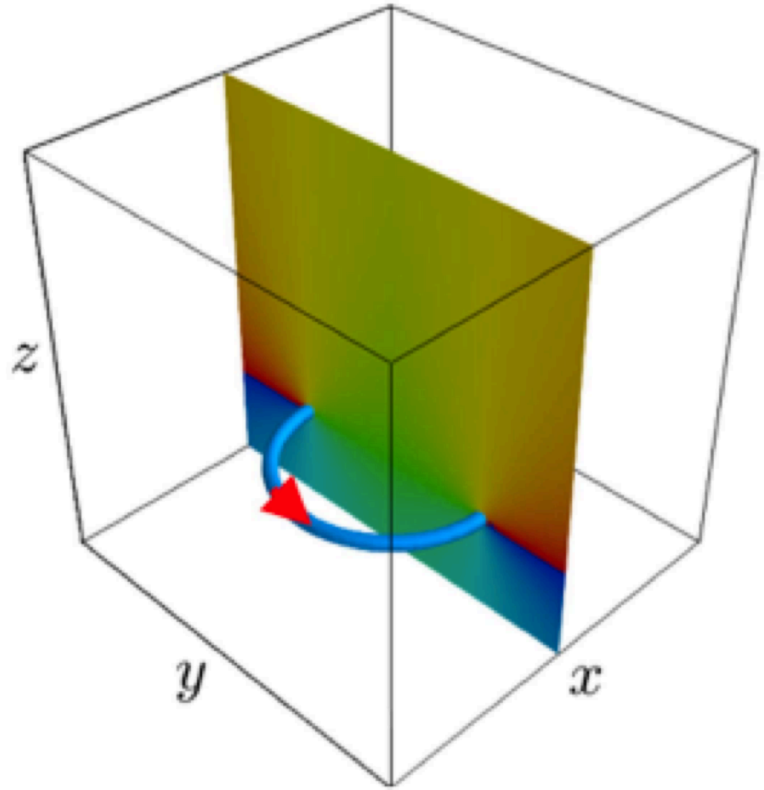
$$\mathbf{u} = u_r \hat{\mathbf{e}}_r + u_\phi \hat{\mathbf{e}}_\phi + u_s \hat{\mathbf{t}}$$

twisted surface in space

Physical effects of twist injection

- ***No twist***

Phase contour in the $(y\text{-}z)$***-plane***



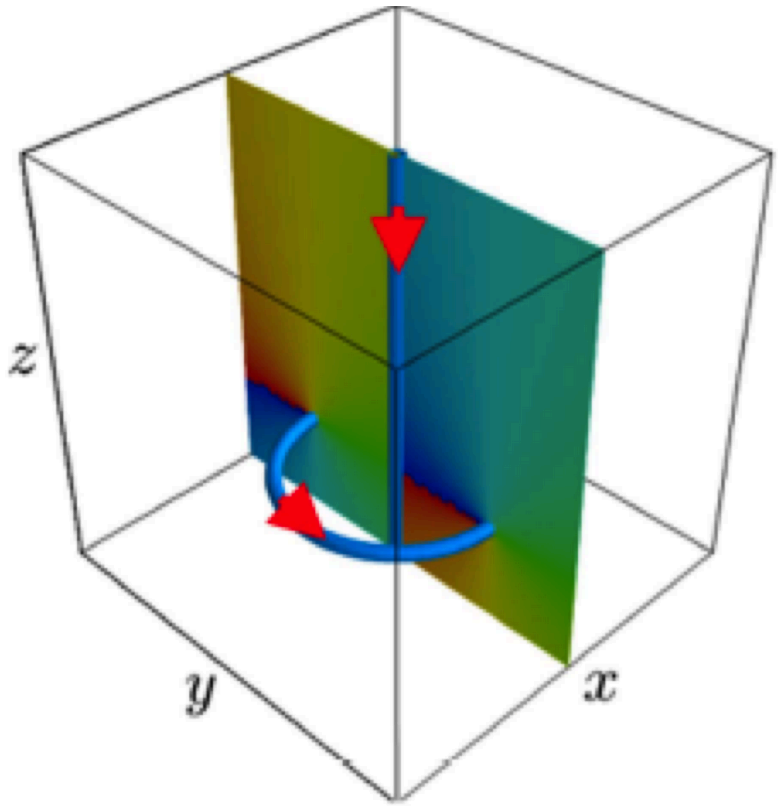
$t = 0$

vortex ring

Physical effects of twist injection

- ***Case A: twist induction***

Phase contour in the $(y\text{-}z)$***-plane***



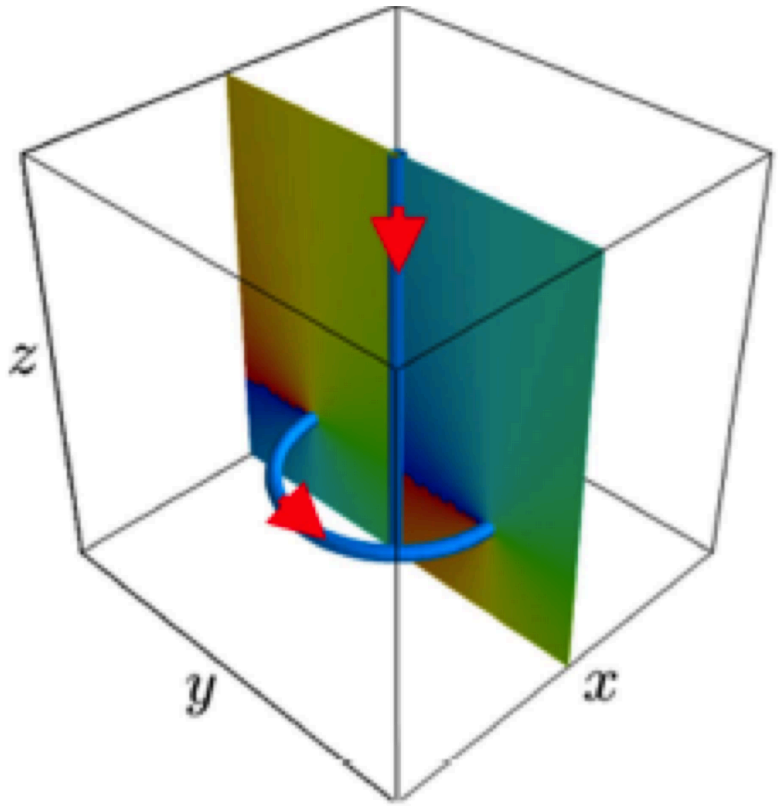
$t = 0$

induction of phase
twist $Tw = 1$ ***on vortex ring***

Physical effects of twist injection

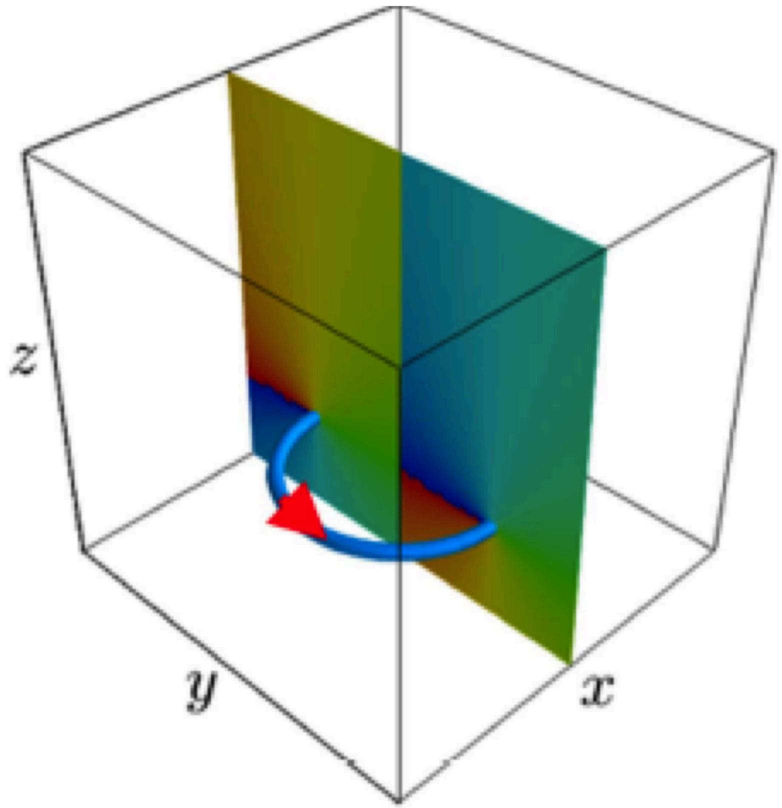
- ***Case A: twist induction***
- ***Case B: twist superimposition***

Phase contour in the $(y\text{-}z)$***-plane***



$t = 0$

induction of phase
twist $Tw = 1$ ***on vortex ring***

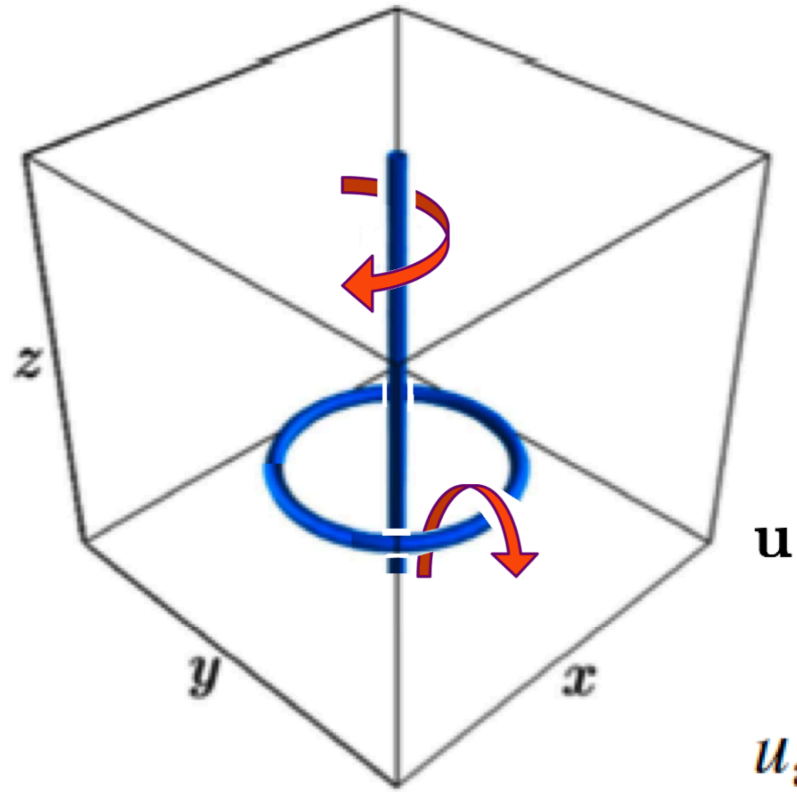


$t = 0$

superimposition of phase
twist $Tw = 1$ ***on vortex ring***

Case A: twist induction (Zucher & Ricca 2018)

induction of phase twist $Tw = 1$ *on vortex ring*



$t = 0$

- ***Biot-Savart induction law:***

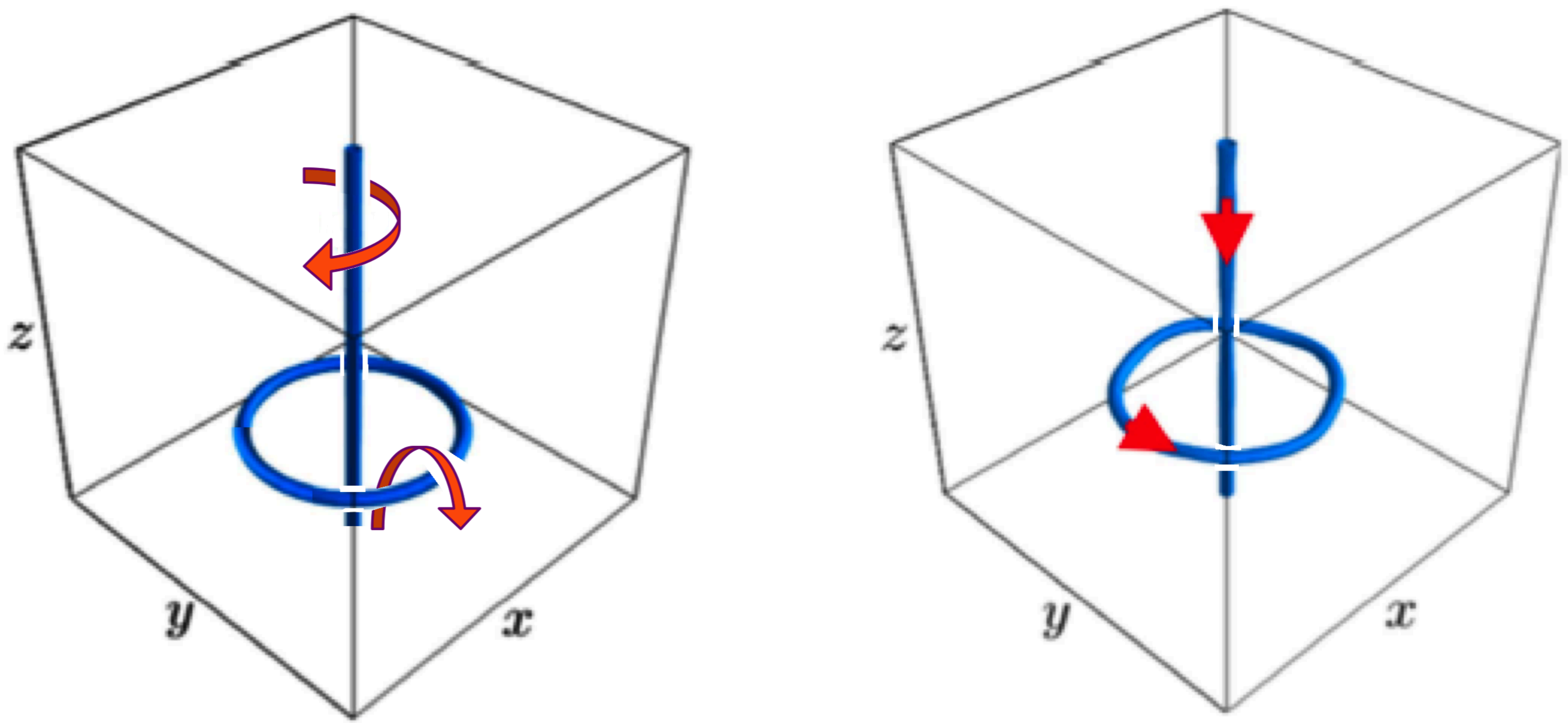
$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi}\int_{V(\mathbf{x}^*)} \frac{\omega(\mathbf{x}^*) \times (\mathbf{x}-\mathbf{x}^*)}{|\mathbf{x}-\mathbf{x}^*|^3}\, d\mathbf{x}^*$$

$$u_{\mathrm{a}} \equiv u_\phi = \frac{\Gamma}{2\pi(R-a)}$$

$$\frac{u_{\mathrm{a}}}{U} = \frac{2}{\left(1-\frac{a}{R}\right)\left(\ln\frac{8R}{a}-\frac{1}{2}\right)} \approx 0.49$$

Case A: twist induction (Zuccher & Ricca 2018)

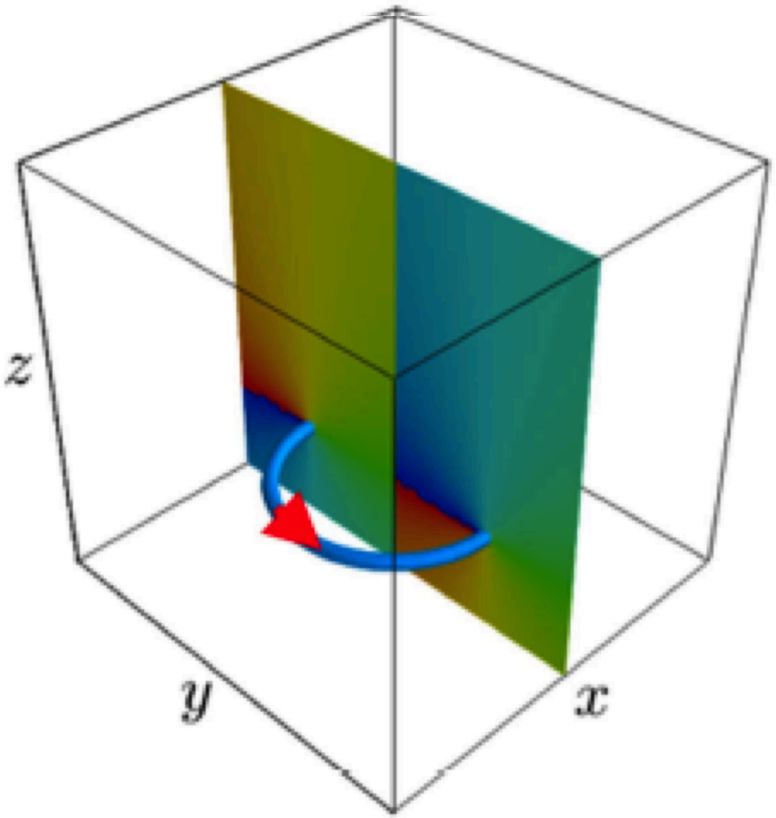
induction of phase twist $Tw = 1$ **on vortex ring**



$t = 0$ $t = 20$

Case B: twist superimposition (Zuccher & Ricca 2018)

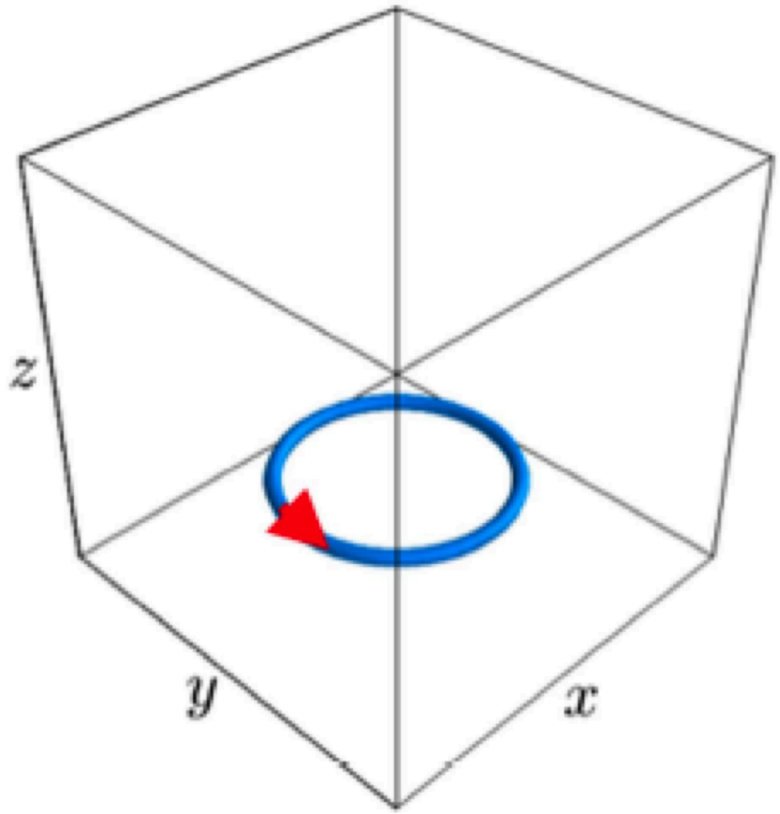
superposition of phase twist $Tw = 1$ on vortex ring



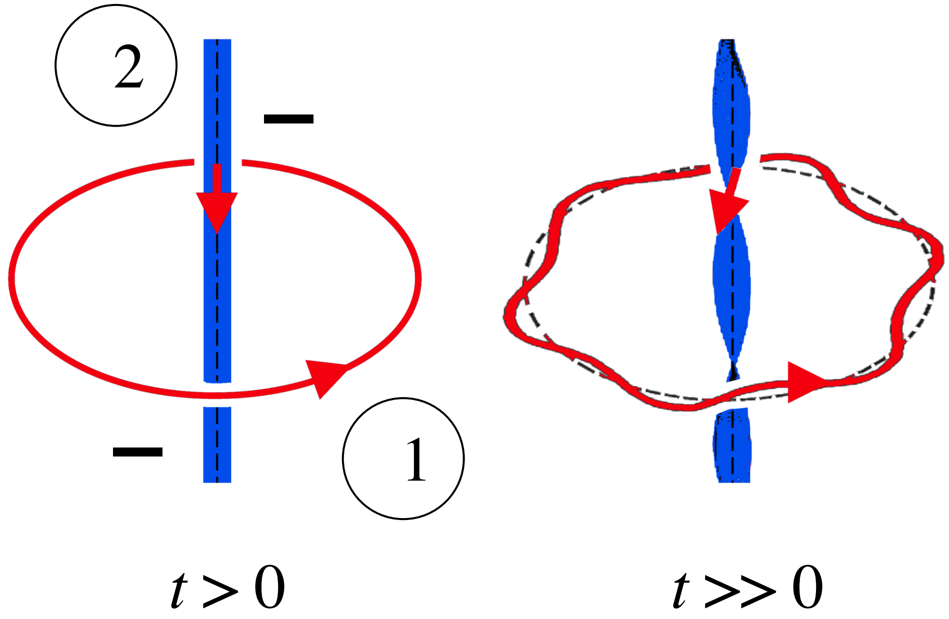
$t = 0$

Case B: twist superimposition (ZucchER & Ricca 2018)

superposition of phase twist $Tw = 1$ *on vortex ring*



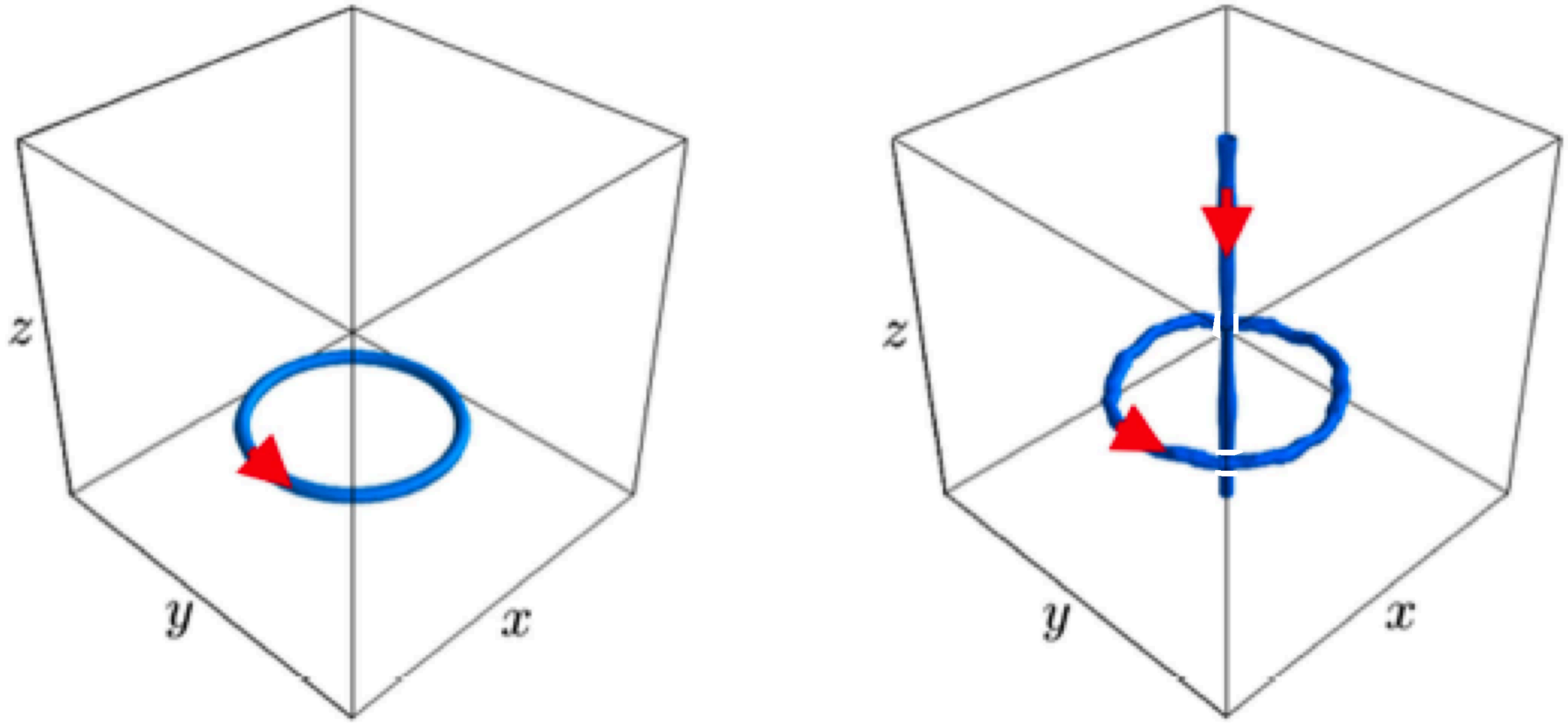
$t = 0$



$$Lk_{\text{tot}} = 0 = Wr_{\text{tot}} + Tw_{\text{tot}} = \underbrace{\cancel{Wr_1}}_{=0} + \underbrace{\cancel{Wr_2}}_{=0} + 2Lk_{12} + Tw_{\text{tot}}$$

Case B: twist superimposition (Zucher & Ricca 2018)

superposition of phase twist** $Tw = 1$ **on vortex ring



$t = 0$ $t = 20$

Generation of new defects by phase twist as Aharonov-Bohm effect!