Universal dynamics for the logarithmic Schrödinger equation
Lecture 3: proof of the main result

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Based on a joint work with
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Some advertisement: the lens transform

Principle : the harmonic oscillator is "nicer" than the free Schrödinger operator (compactness).

The lens transform: if v solves

$$i\partial_t v + \frac{1}{2}\Delta v = \lambda |v|^{4/d} v \quad ; \quad v_{|t=0} = u_0,$$

where $\lambda \in \mathbb{R}$, then u, given for $|t| < \pi/(2\omega)$ by

$$u(t,x) = \frac{1}{(\cos(\omega t))^{d/2}} v\left(\frac{\tan(\omega t)}{\omega}, \frac{x}{\cos(\omega t)}\right) e^{-i\frac{\omega}{2}|x|^2 \tan(\omega t)}$$

solves

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{\omega^2}{2}|x|^2 u + \lambda |u|^{4/d} u \quad ; \quad u_{|t=0} = u_0.$$

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More advertisement: the generalized lens transform

$\Omega \in C(\mathbb{R};\mathbb{R})$:

$$\begin{cases} \ddot{\mu} + \Omega(t)\mu = 0 \quad ; \quad \mu(0) = 0, \quad \dot{\mu}(0) = 1. \\ \ddot{\nu} + \Omega(t)\nu = 0 \quad ; \quad \nu(0) = 1, \quad \dot{\nu}(0) = 0. \end{cases}$$

Let v solve

$$i\partial_t v + \frac{1}{2}\Delta v = H(t)|v|^{2\sigma} v \quad ; \quad v_{|t=0} = u_0.$$

There exists $T > 0$ such that the following holds. Define u by

$$u(t,x) = \frac{1}{\nu(t)^{d/2}} v\left(\frac{\mu(t)}{\nu(t)}, \frac{x}{\nu(t)}\right) e^{i\frac{\dot{\nu}(t)}{\nu(t)}\frac{|x|^2}{2}}, \quad |t| \leqslant T.$$

Then for $|t| < \mu(T)/\nu(T)$, u solves

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{1}{2}\Omega(t)|x|^2 u + h(t)|u|^{2\sigma} u \quad ; \quad u_{|t=0} = u_0,$$

where $h(t) = \nu(t)^{d\sigma-2} H\left(\mu(t)/\nu(t)\right)$.

Logarithmic nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0.$$

In this lecture, we assume $\lambda > 0$ (always).

$\rightsquigarrow$ Formal conservations:

- Mass: $M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^d)}$.
- Energy (Hamiltonian):

$$E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \ln |u(t,x)|^2 dx.$$

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Forget all about scaling

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u.$$

For $k > 0$, ku solves

$$i\partial_t(ku) + \frac{1}{2}\Delta(ku) = \lambda \ln\left(|u|^2\right) ku = \lambda \ln\left(|ku|^2\right) ku - \lambda\left(\ln k^2\right) ku.$$

Gauge transform: $(ku)e^{2it\lambda \ln k}$ solves the same equation as u.

$\rightsquigarrow$ A scaling factor does not change the dynamics.

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Universal dispersion

Lemma

Consider the ODE:

$$\ddot{\tau} = \frac{2\lambda}{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0.$$

It has a unique solution $\tau \in C^2(0,\infty)$*, and, as* $t \to \infty$*,*

$$\tau(t) = 2t\sqrt{\lambda \ln t}\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big), \quad \dot{\tau}(t) = 2\sqrt{\lambda \ln t}\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big),$$

where $\ell(t) := \dfrac{\ln \ln t}{\ln t}$.

Theorem

Let $u_0 \in H^1 \cap \mathcal{F}(H^1)$, *and* $\gamma(y) = e^{-|y|^2/2}$. *Define*

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right) \frac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}}.$$

$$\int_{\mathbb{R}^d} \left(1 + |y|^2 + \left|\ln |v(t,y)|^2\right|\right) |v(t,y)|^2 dy + \frac{1}{\tau(t)^2}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)} \leqslant C.$$

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) dy.$$

$$|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \textit{weakly in } L^1(\mathbb{R}^d).$$

Positive Sobolev norms

Corollary

Let $u_0 \in H^1 \cap \mathcal{F}(H^1) \setminus \{0\}$, *and* $0 < s \leqslant 1$. *As* $t \to \infty$,

$$(\ln t)^{s/2} \lesssim \|u(t)\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim (\ln t)^{s/2},$$

where $\dot{H}^s(\mathbb{R}^d)$ *denotes the standard homogeneous Sobolev space.*

Proof in the case $s = 1$.

$$\nabla u(t,x) = \frac{1}{\tau(t)^{d/2}} \nabla_x \left(v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}} \right)$$

$$= \underbrace{\frac{1}{\tau(t)}\frac{1}{\tau(t)^{d/2}} \nabla_y v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}}}_{\|\cdot\|_{L^2} = \frac{1}{\tau}\|\nabla v\|_{L^2} = \mathcal{O}(1).} + \underbrace{i\dot\tau \frac{1}{\tau(t)^{d/2}} \frac{x}{\tau} v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}}}_{\|\cdot\|_{L^2} = \dot\tau\|yv\|_{L^2} \sim \dot\tau\|y\gamma\|_{L^2} \approx \sqrt{\ln t}}.$$

□

Outline of the proof

- Gaussian case: reduction to ODEs. Appearance of τ.
- This suggests the new unknown function v, which solves

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2, \quad v_{|t=0} = u_0.$$

- A priori estimates for v: directly from the above equation, and from the a priori estimates for u.
- Proof of $|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2$:
  - Polar decomposition (Madelung transform) $\rightsquigarrow$ hyperbolic system (compressible Euler).
  - Change in time variable $\rightsquigarrow$ perturbation of Fokker–Planck equation.
  - A priori estimates $\rightsquigarrow$ weak limit satisfies Fokker–Planck equation.
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where τ is the solution to $\ddot\tau=\frac{2\lambda}{\tau},\ \tau(0)=1,\ \dot\tau(0)=0.$ Then v solves

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$$E(t) := \operatorname{Im}\int_{\mathbb{R}^d} \bar v(t,y)\partial_t v(t,y)dy = E_{\rm kin}(t) + \lambda E_{\rm ent}(t),$$

where

$$E_{\rm kin}(t) := \frac{1}{2\tau(t)^2}\|\nabla_y v(t)\|_{L^2}^2, \quad E_{\rm ent}(t) := \int_{\mathbb{R}^d} |v(t,y)|^2 \ln\left|\frac{v(t,y)}{\gamma(y)}\right|^2 dy.$$

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It satisfies: $\dot E=-2\frac{\dot\tau}{\tau}E_{\rm kin}.$

Csiszár-Kullback inequality

Csiszár-Kullback inequality: if $f, g \geqslant 0$ with $\int f = \int g$, then

$$\|f - g\|_{L^1}^2 \lesssim \|f\|_{L^1} \int f \ln \left(\frac{f}{g}\right) dx.$$

("Algebraic" proof:

$$\forall w \geqslant 0, \quad 3(w-1)^2 \leqslant (2w+4)\left(w \ln w - w + 1\right),$$

$w = f/g$, and Cauchy–Schwarz.)
Therefore,

$$\left\| |v(t)|^2 - \gamma^2 \right\|_{L^1}^2 \lesssim E_{\mathrm{ent}}(t) := \int_{\mathbb{R}^d} |v(t,y)|^2 \ln \left| \frac{v(t,y)}{\gamma(y)} \right|^2 dy,$$

since by definition, $\|v_0\|_{L^2} = \|\gamma\|_{L^2}$. Therefore, $E(t) \geqslant 0$ for all $t \geqslant 0$.

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Lemma

Let $u_0 \in H^1 \cap \mathcal{F}(H^1)$. There exists C such that for all $t \geqslant 0$,

$$\int_{\mathbb{R}^d} \left(1 + |y|^2 + \left|\ln |v(t,y)|^2\right|\right) |v(t,y)|^2 dy + \frac{1}{\tau(t)^2} \|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)} \leqslant C.$$

In addition, $\displaystyle\int_0^\infty \frac{\dot\tau(t)}{\tau(t)^3} \|\nabla_y v(t)\|^2_{L^2} dt < \infty.$

Proof.

$$E_+ := E_{\rm kin} + \lambda \int_{|v|>1} |v|^2 \ln |v|^2 + \lambda \int_{\mathbb{R}^d} |y|^2 |v|^2 \leqslant E(0) + \lambda \int_{|v|<1} |v|^2 \ln \frac{1}{|v|^2}.$$

$$\int_{|v|<1} |v|^2 \ln \frac{1}{|v|^2} \lesssim \int_{\mathbb{R}^d} |v|^{2-\varepsilon} \lesssim \|v\|_{L^2}^{2-(1+d/2)\varepsilon} \|yv\|_{L^2}^{d\varepsilon/2},$$

hence $E_+ \lesssim 1 + E_+^{d\varepsilon/4}$. $\square$

Lemma

Let $u_0 \in H^1 \cap \mathcal{F}(H^1)$. *There exists* C *such that for all* $t \geqslant 0$,

$$\int_{\mathbb{R}^d} \left(1 + |y|^2 + \left|\ln |v(t,y)|^2\right|\right) |v(t,y)|^2 dy + \frac{1}{\tau(t)^2} \|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)} \leqslant C.$$

In addition, $\displaystyle\int_0^\infty \frac{\dot{\tau}(t)}{\tau(t)^3} \|\nabla_y v(t)\|^2_{L^2} dt < \infty.$

Proof.

$$E_+ := E_{\rm kin} + \lambda \int_{|v|>1} |v|^2 \ln |v|^2 + \lambda \int_{\mathbb{R}^d} |y|^2 |v|^2 \leqslant E(0) + \lambda \int_{|v|<1} |v|^2 \ln \frac{1}{|v|^2}.$$

$$\int_{|v|<1} |v|^2 \ln \frac{1}{|v|^2} \lesssim \int_{\mathbb{R}^d} |v|^{2-\varepsilon} \lesssim \|v\|_{L^2}^{2-(1+d/2)\varepsilon} \|yv\|_{L^2}^{d\varepsilon/2},$$

hence $E_+ \lesssim 1 + E_+^{d\varepsilon/4}$. □

Estimates of quadratic observables: order one

$$I_1(t) := \mathrm{Im}\int_{\mathbb{R}^d} \overline{v}(t,y)\nabla_y v(t,y)dy\,, \quad I_2(t) := \int_{\mathbb{R}^d} y|v(t,y)|^2 dy\,.$$

We compute:

$$\dot{I}_1 = -2\lambda I_2\,, \qquad \dot{I}_2 = \frac{1}{\tau(t)^2} I_1\,.$$

Set $\tilde{I}_2 = \tau I_2$: $\ddot{\tilde{I}}_2 = 0$, hence (unless $I_1(0) = 0$)

$$I_2(t) = \frac{1}{\tau(t)}\left(\dot{\tilde{I}}_2(0)t + \tilde{I}_2(0)\right) = \frac{1}{\tau(t)}\left(-I_1(0)t + I_2(0)\right) \underset{t\to\infty}{\sim} \frac{c}{\sqrt{\ln t}}\,,$$

and

$$I_1(t) \underset{t\to\infty}{\sim} \tilde{c}\frac{t}{\sqrt{\ln t}} : \quad v \text{ is oscillatory (in general).}$$

In particular, $\displaystyle\int_{\mathbb{R}^d} y|v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} 0 = \int_{\mathbb{R}^d} y\gamma(y)^2 dy.$

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Estimates of quadratic observables: order two

Introduce $J = \operatorname{Im} \int v\, y \cdot \nabla_y \bar{v}$. Cauchy-Schwarz:

$|J| \leqslant \|yv\|_{L^2} \|\nabla v\|_{L^2} \lesssim \tau(t)$ (previous lemma).

Use the conservation of the energy of u:

$$\frac{d}{dt}\left(E_{\mathrm{kin}} + \frac{(\dot\tau)^2}{2}\int |y|^2|v|^2 - \frac{\dot\tau}{\tau}J + \lambda \int |v|^2 \ln |v|^2 - \lambda d \ln \tau \int |v|^2 + 2\lambda \|\gamma\|_{L^2}^2 \ln\left(\frac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}}\right)\right) = 0.$$

$$\Rightarrow \frac{(\dot\tau)^2}{2}\int |y|^2|v|^2 - \lambda d \ln \tau \int |v|^2 = \mathcal{O}(\dot\tau).$$

But $(\dot\tau)^2 = 2\lambda \ln \tau$ and $\|v\|_{L^2}^2 = \|\gamma\|_{L^2}^2 = \frac{2}{d}\|y\gamma\|_{L^2}^2$.

$$\Rightarrow \|yv(t)\|_{L^2}^2 - \|y\gamma\|_{L^2}^2 = \mathcal{O}\left(\frac{1}{\sqrt{\ln t}}\right).$$

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Madelung transform: Schrödinger meets Euler

Polar decomposition: $v(t,y) = a(t,y)e^{i\phi(t,y)}$, with $a, \phi \in \mathbb{R}$.

$$\rightsquigarrow \begin{cases} \partial_t \phi + \dfrac{1}{2\tau^2}|\nabla_y \phi|^2 + \lambda \ln \left|\dfrac{a}{\gamma}\right|^2 = \dfrac{1}{2\tau^2}\dfrac{\Delta_y a}{a}, \\ \partial_t a + \dfrac{1}{\tau^2}\nabla_y \phi \cdot \nabla_y a + \dfrac{1}{2\tau^2} a \Delta_y \phi = 0\,. \end{cases}$$

Hydrodynamical variables: $\rho = a^2$, $J = a^2 \nabla \phi$.

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2}\nabla \cdot J = 0, \\ \partial_t J + \lambda \nabla \rho + 2\lambda y \rho = \dfrac{1}{4\tau^2}\Delta \nabla \rho - \dfrac{1}{\tau^2}\nabla \cdot \mathrm{Re}\left(\nabla v \otimes \nabla \bar{v}\right). \end{cases}$$

Direct approach. No need to define ϕ: set $\rho = |v|^2$, $J = \mathrm{Im}\, \bar{v} \nabla v$.

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Heuristics

Baby model:

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In terms of ρ only: $\partial_t \left(\tau^2 \partial_t \rho \right) = \lambda \nabla \cdot (\nabla + 2y)\, \rho =: \lambda L \rho$.

Note that $\tau^2 \ll (\dot\tau\tau)^2$: define s such that $\dfrac{\dot\tau\tau}{\lambda}\partial_t = \partial_s$,

$$s = \int \frac{\lambda}{\dot\tau\tau} = \int \frac{\ddot\tau}{2\dot\tau} = \frac{1}{2} \ln \dot\tau(t)\,.$$

Notice that

$$s \sim \frac{1}{4} \ln \ln t\,, \quad t \to \infty\,.$$

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This is good news

Recall that $\rho(t,y) = |v(t,y)|^2$: logarithmic convergence in time,

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \rho(t,y)dy = \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y)dy + \mathcal{O}\left(\frac{1}{\sqrt{\ln t}}\right).$$

We have derived formally:

$$\partial_s \rho = L\rho, \quad L = \nabla \cdot (\nabla + 2y).$$

For such Fokker–Planck equation, convergence to equilibrium with an exponential rate (spectral gap),

$$\|\rho(s) - \gamma^2\|_{L^1} \lesssim e^{-Cs}\|\rho_0 - \gamma^2\|_{L^1}.$$

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Weak limits

Back to the complete the hydrodynamical system:

$$\begin{cases} \partial_t \rho + \frac{1}{\tau^2} \nabla \cdot J = 0, \\ \partial_t J + \lambda \nabla \rho + 2\lambda y \rho = \frac{1}{4\tau^2} \Delta \nabla \rho - \frac{1}{\tau^2} \nabla \cdot \mathrm{Re} \left(\nabla v \otimes \nabla \bar{v} \right). \end{cases}$$

Eliminate J, and introduce the time variable s, $\tilde{\rho}(s, y) := \rho(t, y)$:

$$\partial_s \tilde{\rho} - \frac{2\lambda}{(\dot{\tau})^2} \partial_s \tilde{\rho} + \frac{\lambda}{(\dot{\tau})^2} \partial_s^2 \tilde{\rho} = L\tilde{\rho} - \frac{1}{4\lambda \tau^2} \Delta^2 \tilde{\rho} - \frac{1}{\tau^2} \nabla \cdot \nabla \cdot \mathrm{Re} \left(\nabla v \otimes \nabla \bar{v} \right).$$

For $s \in [-1, 2]$ and $s_n \to \infty$, set $\tilde{\rho}_n(s, y) = \tilde{\rho}(s + s_n, y)$.

De la Vallée-Poussin+ Dunford–Pettis yields (up to a subsequence) $\tilde{\rho}_n \rightharpoonup \tilde{\rho}_\infty$ in $L^p_s(-1, 2; L^1_y)$, for all $p \in [1, \infty)$:

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Liouville property

$$\begin{cases} \partial_s \tilde{\rho} + \dfrac{\dot{\tau}}{\lambda\tau} \nabla \cdot \tilde{J} = 0, \\ \partial_s \tilde{J} + \tau\dot{\tau}\left(\nabla + 2y\right)\tilde{\rho} - \dfrac{\dot{\tau}}{4\lambda\tau}\nabla\Delta\tilde{\rho} = -\dfrac{\dot{\tau}}{\lambda\tau}\nabla \cdot \mathrm{Re}\left(\nabla\tilde{v} \otimes \nabla\bar{\tilde{v}}\right). \end{cases}$$

Since $J = \mathrm{Im}\, \bar{v}\nabla_y v$, a priori estimates on v yield

$$\frac{\dot{\tau}}{\tau}\tilde{J} \in L^2_s L^1_y, \quad \text{hence } \frac{\dot{\tau}}{\tau}\nabla \cdot \tilde{J}_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(-1,2;W^{-1,1}).$$

Therefore, $\partial_s \tilde{\rho}_\infty = 0$.

Conclusion

It is known (A. Arnold, P. Markowich, G. Toscani & A. Unterreiner '01) that any solution to

$$\partial_s \tilde{\rho}_\infty = L\tilde{\rho}_\infty$$

satisfying the above a priori estimates (it is tight) converges for large time

$$\lim_{s\to\infty} \|\tilde{\rho}_\infty(s) - \gamma^2\|_{L^1(\mathbb{R}^d)} = 0.$$

On the other hand, the Liouville property yields $\partial_s \tilde{\rho}_\infty = 0$, hence $\tilde{\rho}_\infty = \gamma^2$. Thus, the limit is unique, and no extraction is needed:

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Growth of Sobolev norms: intermediary indices

Corollary

Let $u_0 \in H^1 \cap \mathcal{F}(H^1) \setminus \{0\}$, and $0 < s \leqslant 1$. As $t \to \infty$,

$$(\ln t)^{s/2} \lesssim \|u(t)\|_{\dot H^s(\mathbb{R}^d)} \lesssim (\ln t)^{s/2},$$

where $\dot H^s(\mathbb{R}^d)$ denotes the standard homogeneous Sobolev space.

$s = 1$: direct consequence of a priori estimates, and the convergence of the second momentum of v,

$$\nabla u(t,x) = \frac{1}{\tau(t)^{d/2}} \nabla_x \left(v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}} \right)$$

$$= \underbrace{\frac{1}{\tau(t)} \frac{1}{\tau(t)^{d/2}} \nabla_y v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}}}_{\|\cdot\|_{L^2} = \frac{1}{\tau}\|\nabla v\|_{L^2} = \mathcal{O}(1).} + \underbrace{i\dot\tau \frac{1}{\tau(t)^{d/2}} \frac{x}{\tau} v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}}}_{\|\cdot\|_{L^2} = \dot\tau\|yv\|_{L^2} \sim \dot\tau\|y\gamma\|_{L^2} \approx \sqrt{\ln t}}.$$

By interpolation, for $0 < s < 1$, we readily have

$$\|u(t)\|_{\dot{H}^s} \lesssim \|u(t)\|_{L^2}^{1-s} \|u(t)\|_{\dot{H}^1}^{s} \lesssim (\ln t)^{s/2}\,.$$

For the other (more interesting) estimate, we use the property

$$\int |y|^{2s}|v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int |y|^{2s}\gamma^2(y)dy.$$

It is not obvious, and uses all the conclusions of the main theorem (including the weak convergence). Assuming this property:

Lemma (with T. Alazard, '07)

If $u \in H^1(\mathbb{R}^d)$ and w is such that $\nabla w \in L^\infty(\mathbb{R}^d)$,

$$\| |w|^s u\|_{L^2} \leqslant \|u\|_{\dot{H}^s} + \|(\nabla - iw)u\|_{L^2}^{s} \|u\|_{L^2}^{1-s} + C\left(1 + \|\nabla w\|_{L^\infty}\right)\|u\|_{L^2}.$$

Application: $w(t,x) = \dfrac{\dot{\tau}(t)}{\tau(t)}x$.

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It is not obvious, and uses all the conclusions of the main theorem (including the weak convergence). Assuming this property:

Lemma (with T. Alazard, '07)

If $u \in H^1(\mathbb{R}^d)$ and w is such that $\nabla w \in L^\infty(\mathbb{R}^d)$,

$$\| |w|^s u\|_{L^2} \leqslant \|u\|_{\dot H^s} + \|(\nabla - iw)u\|_{L^2}^s \|u\|_{L^2}^{1-s} + C\left(1 + \|\nabla w\|_{L^\infty}\right)\|u\|_{L^2}.$$

Application: $w(t,x) = \dfrac{\dot\tau(t)}{\tau(t)} x.$

By interpolation, for $0 < s < 1$, we readily have

$$\|u(t)\|_{\dot{H}^s} \lesssim \|u(t)\|_{L^2}^{1-s}\|u(t)\|_{\dot{H}^1}^{s} \lesssim (\ln t)^{s/2}\,.$$

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Convergence of fractional momenta

Main theorem:

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) dy, \quad |v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \text{w-}L^1.$$

This implies that $|v(t,\cdot)|^2$ converges to γ^2 in Wasserstein distance:

$$W_2\left(\frac{|v(t,\cdot)|^2}{\pi^{d/2}}, \frac{\gamma^2}{\pi^{d/2}}\right) \underset{t\to\infty}{\longrightarrow} 0,$$

where, for ν_1 and ν_2 probability measures,

$$W_p(\nu_1,\nu_2) = \inf\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\mu(x,y)\right)^{1/p}; \quad (\pi_j)_\sharp \mu = \nu_j\right\},$$

where μ varies among all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$, and $\pi_j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ denotes the canonical projection onto the j-th factor.

A convenient property of Wasserstein distance

We find (e.g. in C. Villani's book): if

$$W_2\left(\nu(t), \nu_\infty\right) \underset{t\to\infty}{\longrightarrow} 0,$$

then for $0 < s < 1$,

$$\int |x|^{2s}\nu(t,dx) \underset{t\to\infty}{\longrightarrow} \int |x|^{2s}\nu_\infty(dx).$$

Extensions and open questions

- Compressible fluid mechanics:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \dfrac{\varepsilon^2}{2} \rho \nabla \left(\dfrac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + \nu \operatorname{div}(\rho D u), \end{cases}$$

  with $\varepsilon, \nu \geqslant 0$, P convex and $P'(0) > 0$. Generalized isothermal fluid.
  Joint work with Kleber Carrapatoso & Matthieu Hillairet.
- Vlasov equation (fusion plasmas):

$$\partial_t f + \xi \partial_x f - (\partial_x \ln(\rho)) \partial_\xi f = 0, \quad \rho(t,x) = \int_{\mathbb{R}} f(t,x,\xi) d\xi.$$

  Partial results with Anne Nouri.
  Work in progress by Guillaume Ferrière.
- Dynamics for logNLS in the case $\lambda < 0$: numerical simulations with Weizhu Bao, Chunmei Su & Qinglin Tang.
  Analytical aspects: PhD thesis of Guillaume Ferrière.