Universal dynamics for the logarithmic Schrödinger equation
Lecture 2: Cauchy problem and the Gaussian case

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Based on a joint work with
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Logarithmic nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0.$$

⇝ Formal conservations:

- Mass: $M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^d)}$.
- Energy (Hamiltonian):
  $E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \ln |u(t,x)|^2 dx.$

⇝ Mathematical study:

$$W := \left\{ u \in H^1(\mathbb{R}^d)\,,\, x \mapsto |u(x)|^2 \ln |u(x)|^2 \in L^1(\mathbb{R}^d) \right\}.$$

Theorem (Th. Cazenave & A. Haraux '80)

$\lambda < 0$, $u_0 \in W$: *unique, global solution* $u \in C(\mathbb{R}; W)$. *The mass* $M(u)$ *and the energy* $E(u)$ *are independent of time.*

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Logarithmic NLS: "defocusing" case

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0.$$

$\rightsquigarrow$ In the case $\lambda > 0$, global Cauchy problem less studied.

$$\mathcal{F}(H^\alpha) = \left\{ u \in L^2(\mathbb{R}^d)\,,\, x \mapsto \langle x\rangle^\alpha u(x) \in L^2(\mathbb{R}^d) \right\},$$

Theorem

$\lambda > 0$, $u_0 \in \mathcal{F}(H^\alpha) \cap H^1(\mathbb{R}^d)$ *with* $0 < \alpha \leqslant 1$.
There exists a unique, global solution $u \in L^\infty_{\rm loc}(\mathbb{R}; \mathcal{F}(H^\alpha) \cap H^1)$.
Mass $M(u)$ *and energy* $E(u)$ *are independent of time.*
If in addition $u_0 \in H^2(\mathbb{R}^d)$, *then* $u \in L^\infty_{\rm loc}(\mathbb{R}; H^2)$.

Remark

$\mathcal{F}(H^\alpha) \cap H^1 \subsetneq W$. Issue $= \{|u| < 1\}$.

On the functional space

$$E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \ln |u(t,x)|^2 dx.$$

$H^1(\mathbb{R}^d)$ is rather natural (even though one might simply expect $L^2(\mathbb{R}^d)$).

Recall that from a priori estimates, the use of a momentum is rather natural:

$$0 \leqslant E_+(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{|u|>1} |u(t,x)|^2 \ln |u(t,x)|^2 dx$$
$$\leqslant E(u_0) + \lambda \int_{|u|<1} |u(t,x)|^2 \ln \frac{1}{|u(t,x)|^2} dx.$$

The negative term is controlled thanks to

$$0 \leqslant \int_{|u|<1} |u(t,x)|^2 \ln \frac{1}{|u(t,x)|^2} dx \leqslant C_\varepsilon \int_{|u|<1} |u(t,x)|^{2-\varepsilon} dx,$$

which in turn is controlled by a $\mathcal{F}(H^s)$-norm for $s \geqslant s(\varepsilon) > 0$, with $s(\varepsilon) \to 0$ as $\varepsilon \to 0$.

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Uniqueness

u_1 and u_2 two solutions: $u := u_1 - u_2$ satisfies

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda\Big(\ln\left(|u_1|^2\right) u_1 - \ln\left(|u_2|^2\right) u_2\Big).$$

Energy estimate:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^d)} = \lambda \operatorname{Im}\int_{\mathbb{R}^d} \Big(\ln\left(|u_1|^2\right) u_1 - \ln\left(|u_2|^2\right) u_2\Big)(\bar{u}_1 - \bar{u}_2)(t)dx.$$

Lemma (Cazenave–Haraux '80)

$$\left|\operatorname{Im}\left(\left(z_2 \ln|z_2|^2 - z_1 \ln|z_1|^2\right)\left(\bar{z}_2 - \bar{z}_1\right)\right)\right| \leqslant 4|z_2 - z_1|^2, \quad \forall z_1, z_2 \in \mathbb{C}.$$

We infer $\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^d)} \leqslant 4\lambda\|u(t)\|^2_{L^2(\mathbb{R}^d)}$: Gronwall.

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Uniqueness: an argument was missing

What regularity allows to claim uniqueness?

Uniqueness is claimed in the class

$$u \in L^\infty_{\rm loc}(\mathbb{R};\mathcal{F}(H^\alpha)\cap H^1).$$

This is not enough to make sense of the initial data in $L^2(\mathbb{R}^d)$. Following the approach of Th. Cazenave & A. Haraux, we prove that any such solution satisfies $u \in C(\mathbb{R};L^2)$.

- Obviously, $\Delta u \in L^\infty_{\rm loc}(\mathbb{R};H^{-1})$.
- Claim: $u\ln|u|^2 \in L^\infty_{\rm loc}(\mathbb{R};L^2(\mathbb{R}^d))$. Indeed,

$$\int |u|^2(\ln|u|^2)^2 \lesssim \underbrace{\int |u|^{2-\varepsilon}}_{\lesssim \|u\|_{L^2}^{2-\varepsilon-\frac{d\varepsilon}{2\alpha}}\|x^\alpha u\|_{L^2}^{\frac{d\varepsilon}{2\alpha}}} + \underbrace{\int |u|^{2+\varepsilon}}_{\lesssim \|u\|_{L^2}^{2-\varepsilon-d\varepsilon/2}\|\nabla u\|_{L^2}^{d\varepsilon/2}}.$$

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Existence in $H^1 \cap \mathcal{F}(H^s)$

Recall that $f : z \mapsto z \ln |z|^2$ is not Lipschitz continuous.

Cazenave–Haraux '80: approximate f near 0 by its Taylor expansion at $z = \varepsilon$, and compactness arguments, thanks to suitable a priori estimates, in the case $\lambda < 0$.

$\rightsquigarrow$ This approach does not seem to work in the case $\lambda > 0$.

Different strategy: make the nonlinearity locally Lipschitzean and avoid the unboundedness of the logarithm near zero,

$$i\partial_t u_\varepsilon + \frac{1}{2}\Delta u_\varepsilon = \lambda \ln \left(\varepsilon + |u_\varepsilon|^2\right) u_\varepsilon , \quad u_{\varepsilon|t=0} = u_0.$$

$\rightsquigarrow$ For fixed $\varepsilon > 0$: global well-posedness in $L^2(\mathbb{R}^d)$.

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A priori estimates

$$i\partial_t\partial_j u_\varepsilon + \frac{1}{2}\Delta\partial_j u_\varepsilon = \lambda \ln\left(\varepsilon + |u_\varepsilon|^2\right)\partial_j u_\varepsilon + 2\lambda\frac{1}{\varepsilon+|u_\varepsilon|^2}\,\mathrm{Re}(\bar{u}_\varepsilon\partial_j u_\varepsilon)u_\varepsilon.$$

L^2 estimate+Gronwall: $\|u_\varepsilon(t)\|_{H^1} \leqslant \|u_0\|_{H^1}e^{C|t|}$, C independent of ε.

Let $I_{\varepsilon,\alpha}(t) := \int_{\mathbb{R}^d}\langle x\rangle^{2\alpha}|u_\varepsilon|^2(t,x)\,dx$. Energy estimate:

$$\frac{d}{dt}I_{\varepsilon,\alpha}(t) = 2\alpha\,\mathrm{Im}\int\frac{x\cdot\nabla u_\varepsilon}{\langle x\rangle^{2-2\alpha}}\,\overline{u}_\varepsilon(t)dx \leqslant 2\alpha\|\langle x\rangle^{2\alpha-1}u_\varepsilon(t)\|_{L^2}\|\nabla u_\varepsilon(t)\|_{L^2}$$
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since $\alpha \leqslant 1$.

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Convergence of the approximating sequence

$$i\partial_t u_\varepsilon + \frac{1}{2}\Delta u_\varepsilon = \lambda \ln\left(\varepsilon + |u_\varepsilon|^2\right) u_\varepsilon\,, \quad u_{\varepsilon|t=0} = u_0.$$

$(u_\varepsilon)_\varepsilon$ uniformly bounded in $L^\infty((-T,T);H^1\cap\mathcal{F}(H^\alpha))$, for any $T>0$.
Equation $\rightsquigarrow$ time compactness in $H^{-2}(\mathbb{R}^d)$.
Arzela-Ascoli $\rightsquigarrow$ convergence of a subsequence: existence of a global weak solution.

Convergence of the approximating sequence

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Arzela-Ascoli $\rightsquigarrow$ convergence of a subsequence: existence of a global weak solution.

Convergence of the approximating sequence

$$i\partial_t u_\varepsilon + \frac{1}{2}\Delta u_\varepsilon = \lambda \ln\left(\varepsilon + |u_\varepsilon|^2\right) u_\varepsilon \,, \quad u_{\varepsilon|t=0} = u_0.$$

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Strong convergence

• In the case $\lambda < 0$, strong convergence result for another approximating procedure (Cazenave-Haraux '80, M. Hayashi '18),

$$\|u - \tilde{u}_\varepsilon\|_{L^\infty([0,T];W)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

• In the case $\lambda > 0$, the strong convergence

$$\|u - u_\varepsilon\|_{L^\infty([0,T];L^2)} \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \left(\text{hence } \|u - u_\varepsilon\|_{L^\infty([0,T];H^s)} \underset{\varepsilon\to 0}{\longrightarrow} 0,\ s < 1\right)$$

is proved for $d = 1$ and $u_0 \in H^1 \cap \mathcal{F}(H^1)$, and

$$\|u - u_\varepsilon\|_{L^\infty([0,T];H^s)} \underset{\varepsilon\to 0}{\longrightarrow} 0, \quad s < 2,$$

is proved for $d = 1, 2, 3$ and $u_0 \in H^2 \cap \mathcal{F}(H^2)$. See Bao-C.-Su-Tang... and next slide. Key inequality:

$$\frac{d}{dt}\|u(t) - u_\varepsilon(t)\|_{L^2}^2 \leqslant 4|\lambda| \left(\|u(t) - u_\varepsilon(t)\|_{L^2}^2 + \sqrt{\varepsilon}\|u(t) - u_\varepsilon(t)\|_{L^1}\right).$$

Higher regularity

Problem: $z \mapsto z \ln |z|^2$ is not smooth (at the origin). Impossible to differentiate the equation too many times.

Propagation of H^2 regularity: Kato's trick.

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u.$$

$\rightsquigarrow$ To control the H^2-norm, control $\|\partial_t u\|_{L^2}$, since

$$\begin{aligned}\left\|u \ln |u|^2\right\|_{L^2} &\lesssim \left\| |u|^{1+\delta} \right\|_{L^2} + \left\| |u|^{1-\delta} \right\|_{L^2}, \quad \forall \delta > 0,\\ &\lesssim \|u\|_{H^1}^{1+\delta} + \|u\|_{\mathcal{F}(H^\alpha)}^{1-\delta}, \text{ for } 0 < \delta \ll 1.\end{aligned}$$

Uniform control of $\|\partial_t u_\varepsilon\|_{L^2}$: (almost) like the control of $\|\nabla u_\varepsilon\|_{L^2}$.

$\rightsquigarrow$ Open question: what about H^3?

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Gaussian data: general case

Theorem (Gaussian data, $d = 1$)

Suppose $d = 1$, and let

$$u_0(x) = b_0 e^{-a_0 x^2/2}, \quad a_0, b_0 \in \mathbb{C},\ \alpha_0 := \mathrm{Re}\, a_0 > 0.$$

The solution of LogNLS is given by

$$u(t,x) = \frac{b_0}{\sqrt{r(t)}} \exp\left(-i\phi(t) - \alpha_0 \frac{x^2}{2r(t)^2} + i\frac{\dot r(t)}{r(t)}\frac{x^2}{4}\right),$$

where $\phi \in \mathbb{R}$ and $r > 0$ solve the ODEs

$$\dot\phi = \frac{\alpha_0}{r^2} + \lambda \ln |b_0|^2 + \lambda \ln r, \quad \phi(0) = 0,$$

$$\ddot r = \frac{4\alpha_0^2}{r^3} + \frac{4\lambda\alpha_0}{r}, \quad r(0) = 1,\ \dot r(0) = -2\,\mathrm{Im}\, a_0.$$

Multidimensional case: tensorization

Corollary (Gaussian data, $d \geqslant 2$)

If

$$u_0(x_1, \dots, x_d) = \prod_{j=1}^{d} u_{0j}(x_j),$$

with u_{0j} *as in the previous result, then*

$$u(t,x) = \prod_{j=1}^{d} u_j(t, x_j),$$

with u_j *given by the formula of the previous result.*

Remark

This tensorization phenomenon is due to the property $ln|ab| = \ln|a| + \ln|b|$.

Gaussian data: case $\lambda > 0$

Theorem

$\lambda > 0$, $u_0(x) = b_0 e^{-\frac{1}{2}\sum_{j=1}^d a_{0j}x_j^2}$, *with* $b_0, a_{0j} \in \mathbb{C}$, $\alpha_{0j} = \operatorname{Re} a_{0j} > 0$. *Then*

$$u(t,x) = b_0 \prod_{j=1}^d \frac{1}{\sqrt{r_j(t)}} \exp\left(i\phi_j(t) - \alpha_{0j}\frac{x_j^2}{2r_j^2(t)} + i\frac{\dot{r}_j(t)}{r_j(t)}\frac{x_j^2}{2}\right)$$

for some real-valued functions ϕ_j, r_j *depending on* t *only.*

$$r_j(t) = 2t\sqrt{\lambda\alpha_{0j}\ln t}\left(1 + o(1)\right), \quad \dot{r}_j(t) = 2\sqrt{\lambda\alpha_{0j}\ln t}\left(1 + o(1)\right).$$

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \underset{t\to\infty}{\sim} \frac{1}{\left(t\sqrt{\ln t}\right)^{d/2}} \frac{\|u_0\|_{L^2}}{\left(2\lambda\sqrt{2\pi}\right)^{d/2}},$$

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} 2\lambda d\|u_0\|^2_{L^2(\mathbb{R}^d)} \ln t.$$

Three new features

- Dispersion: usual $t^{-d/2}$ rate becomes $(t\sqrt{\ln t})^{-d/2}$.
- Unboundedness in H^1: $\|\nabla u(t)\|_{L^2} \approx \sqrt{\ln t}$.
- *Universal profile:*

$$(2t\sqrt{\lambda \ln t})^{d/2} \left| u\left(t, x \times 2t\sqrt{\lambda \ln t}\right)\right| \underset{t\to\infty}{\longrightarrow} \frac{\|u_0\|_{L^2}}{\pi^{d/4}} e^{-|x|^2/2},$$

regardless of the initial variances.

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Schrödinger equation with quadratic Hamiltonian

Model case: the harmonic oscillator,

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{|x|^2}{2}u \quad ; \quad u_{|t=0} = u_0.$$

- Eigenbasis: Hermite functions, $\sigma_p = \frac{1}{2} + \mathbb{N}$.
- The solutions are periodic in time.
- Explicit fundamental solution: Mehler's formula. For $|t| < \pi/2$,

$$u(t,x) = \frac{1}{(2i\pi \sin t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\sin t}\left(\frac{|x|^2+|y|^2}{2}\cos t - x\cdot y\right)} u_0(y)dy.$$

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Schrödinger equation with quadratic Hamiltonian

When the Hamiltonian depends on time (1D case, not even general):

$$i\partial_t u + \frac{a(t)}{2}\partial_x^2 u = b(t)\frac{x^2}{2}u \quad ; \quad u_{|t=0} = u_0.$$

- Envelope equation in the semi-classical limit of coherent states.
- A generalized Mehler's formula is available.
- If u_0 is Gaussian, then so is $u(t,\cdot)$ for all $t \in \mathbb{R}$ (Hagedorn '80): a PDE becomes a system of ODEs.

Time dependent harmonic oscillator

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \Omega(t)\frac{x^2}{2}u \quad ; \quad u_{|t=0} = u_0.$$

Seek formally the solution as

$$u(t,x) = \frac{1}{(2i\pi\mu(t))^{1/2}}\int_{\mathbb{R}} e^{\frac{i}{2}\left(\alpha(t)x^2+2\beta(t)xy+\gamma(t)y^2\right)}u_0(y)dy.$$

We find:

$$x^2 : \quad \dot\alpha + \alpha^2 + \Omega = 0; \quad xy : \quad \dot\beta + \alpha\beta = 0; \quad y^2 : \quad \dot\gamma + \beta^2 = 0,$$
$$\mathrm{Im}(\mathbb{C}) : \quad \dot\mu = \alpha\mu.$$

μ is given by $\ddot\mu + \Omega(t)\mu = 0 \quad ; \quad \mu(0) = 0, \quad \dot\mu(0) = 1.$

We also have

$$\alpha = \frac{\dot\mu}{\mu}, \quad \beta(t) = \frac{-1}{\mu(t)}, \quad \gamma(t) = \frac{1}{\mu(t)\dot\mu(t)} - \int_0^t \frac{\Omega(\tau)}{(\dot\mu(\tau))^2}d\tau.$$

Examples.

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Examples.

Gaussian case: from PDE to ODE in the linear case

Plug $u(t,x) = b(t)e^{-a(t)x^2/2}$ into the equation

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We find:

$$i\dot{b} - i\dot{a}\frac{x^2}{2}b - \frac{ab}{2} + a^2\frac{x^2}{2}b = \Omega\frac{x^2}{2}b,$$

hence

$$i\dot{a} - a^2 + \Omega = 0\,; \quad i\dot{b} - \frac{ab}{2} = 0.$$

We can express b as a function of a:

$$b(t) = b_0 e^{-\frac{i}{2}\int_0^t a},$$

and a solves a complex Riccati equation,

$$i\dot{a} - a^2 = \Omega, \quad a_{|t=0} = a_0.$$

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The complex Riccati equation

$$i\dot{a} - a^2 = \Omega, \quad a_{|t=0} = a_0.$$

Seeking a of the form $a = -i\dfrac{\dot{\omega}}{\omega}$, we get: $\ddot{\omega} + \Omega\omega = 0$.
$\rightsquigarrow$ Same linear ODE as in the computation of Mehler's formula.

Amplitude (dispersion or not):

$$b(t) = b_0 e^{-\frac{i}{2}\int_0^t a} = b_0 e^{-\frac{1}{2}\int_0^t \frac{\dot{\omega}}{\omega}} = \frac{b_0}{\sqrt{\omega(t)}}.$$

Gaussian case: from PDE to ODE for logNLS

Suppose $d = 1$, and plug $u(t,x) = b(t)e^{-a(t)x^2/2}$ into the equation:

$$i\dot{b} - i\dot{a}\frac{x^2}{2}b - \frac{ab}{2} + a^2\frac{x^2}{2}b = \lambda\left(\ln\left(|b|^2\right) - (\mathrm{Re}\, a)\, x^2\right) b,$$

hence

$$i\dot{a} - a^2 = 2\lambda\,\mathrm{Re}\, a\,; \quad i\dot{b} - \frac{ab}{2} = \lambda b \ln\left(|b|^2\right).$$

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$$b(t) = b_0 \exp\left(-i\lambda t \ln\left(|b_0|^2\right) - \frac{i}{2}A(t) - i\lambda\,\mathrm{Im}\int_0^t A(s)sds\right),$$

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Toward a universal ODE

$$i\dot{a} - a^2 = 2\lambda \,\mathrm{Re}\, a, \quad a_{|t=0} = a_0 = \alpha_0 + i\beta_0.$$

We seek a of the form $a = -i\dfrac{\dot\omega}{\omega}$. We get: $\ddot\omega = 2\lambda\omega \,\mathrm{Im}\, \dfrac{\dot\omega}{\omega}$.

Polar decomposition: $\omega = re^{i\theta}$,

$$\ddot r - (\dot\theta)^2 r = 2\lambda r\dot\theta\,; \quad \ddot\theta r + 2\dot\theta\dot r = 0.$$

Notice that

$$\dot\theta_{|t=0} = \alpha_0\,, \quad \left(\frac{\dot r}{r}\right)_{|t=0} = -\beta_0\,.$$

We decide $r(0) = 1$ so $\dot\theta(0) = \mathrm{Re}\, a_0 = \alpha_0$ and $\dot r(0) = -\,\mathrm{Im}\, a_0 = -\beta_0$. Note

$$\frac{d}{dt}\left(r^2\dot\theta\right) = r\left(2\dot r\dot\theta + r\ddot\theta\right) = 0\,,$$

and we can express the problem in terms of r only:

$$a(t) = \frac{\alpha_0}{r(t)^2} - i\frac{\dot r(t)}{r(t)}, \quad \ddot r = \frac{\alpha_0^2}{r^3} + 2\lambda\frac{\alpha_0}{r}, \quad r(0) = 1\,, \;\; \dot r(0) = -\beta_0\,.$$

Toward a universal ODE

$$i\dot{a} - a^2 = 2\lambda \operatorname{Re} a, \quad a_{|t=0} = a_0 = \alpha_0 + i\beta_0.$$

We seek a of the form $a = -i\frac{\dot{\omega}}{\omega}$. We get: $\ddot{\omega} = 2\lambda\omega \operatorname{Im} \frac{\dot{\omega}}{\omega}$.

Polar decomposition: $\omega = re^{i\theta}$,

$$\ddot{r} - (\dot{\theta})^2 r = 2\lambda r\dot{\theta}\,; \quad \ddot{\theta}r + 2\dot{\theta}\dot{r} = 0.$$

Notice that

$$\dot{\theta}_{|t=0} = \alpha_0\,, \quad \left(\frac{\dot{r}}{r}\right)_{|t=0} = -\beta_0\,.$$

We decide $r(0) = 1$ so $\dot{\theta}(0) = \operatorname{Re} a_0 = \alpha_0$ and $\dot{r}(0) = -\operatorname{Im} a_0 = -\beta_0$. Note

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Partial conclusion

The solution is $u(t,x) = b(t)e^{-a(t)x^2/2}$ with

$$b(t) = b_0 \exp\left(-i\lambda t \ln\left(|b_0|^2\right) - \frac{i}{2}A(t) - i\lambda \operatorname{Im}\int_0^t A(s)sds\right),$$

where we have set $A(t) := \int_0^t a(s)ds$, and a is given by

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Remark

A solution to $\ddot\rho = \dfrac{c^4}{\rho^3}$, is given by $\rho(t) = c\sqrt{1+t^2}$: usual dispersion.

The relevant equation is the red one. Multiplying by $\dot r$ and integrating,

$$(\dot r)^2 = C_0 - \frac{\alpha_0^2}{r^2} + 4\lambda\alpha_0 \ln r.$$

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$\lambda < 0$: periodic behavior

$(\dot{r})^2 = C_0 - \frac{\alpha_0^2}{r^2} + 4\lambda\alpha_0 \ln r =: -2U(r).$

ρ	0		$\sqrt{\frac{\alpha_0}{2\lvert\lambda\rvert}}$		$+\infty$
$U'(\rho)$	‖	$-$	0	$+$	
$U(\rho)$	‖	$+\infty$ ↘	$U_{\min}$	↗ $+\infty$	

$$U_{\min} = -\frac{1}{2}\beta_0^2 + \frac{\alpha_0^2}{2}\left(x - 1 - x\ln x\right)\Big|_{x=\frac{2|\lambda|}{\alpha_0}} \leqslant 0,$$

$U_{\min} < 0$ unless $\beta_0 = \dot{\rho}(0) = 0$ and $\alpha_0 = 2|\lambda|$, the only case where $U_{\min} = 0$: Gausson.

$\lambda < 0$: periodic behavior

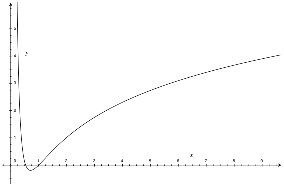

Figure: Potential U in the case $a_0 = 1$ and $\lambda = -1$.

For every energy $E > U_{\min}$, the equation $U(\rho) = E$ has two distinct solutions. We infer that all the solutions are periodic, and the half-period is given by

$$\frac{T}{2} = \int_{\rho_*}^{\rho^*} \frac{d\rho}{\sqrt{E - U(\rho)}},$$

where $\rho_* < \rho^*$ are the two above mentionned solutions.

Case $\lambda > 0$: universal dispersion

$$a(t) = \frac{\alpha_0}{r(t)^2} - i\frac{\dot r(t)}{r(t)}, \quad \ddot r = \frac{\alpha_0^2}{r^3} + 2\lambda\frac{\alpha_0}{r}, \quad r(0) = 1\,, \ \ \dot r(0) = -\beta_0\,.$$

We can prove: for $t \geqslant T$, $\ddot r > 0$, and $r(t) \to \infty$ as $t \to \infty$. Hence

$$\ddot r_{\rm eff} = \frac{2\lambda\alpha_0}{r_{\rm eff}} \quad (\alpha_0 > 0).$$

Up to scaling (and initial data): $\ddot\tau = \dfrac{2\lambda}{\tau}$.

By integration,

$$\dot r_{\rm eff} = \sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\rm eff}},$$

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Universal dispersion

$$\dot{r}_{\rm eff} = \sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\rm eff}}.$$

Separate variables:

$$\int^{r_{\rm eff}} \frac{dz}{\sqrt{C_0 + 4\lambda\alpha_0 \ln z}} = t - T.$$

Set $y = \sqrt{C_0 + 4\lambda\alpha_0 \ln z}$. The left hand side becomes

$$\frac{1}{2\lambda\alpha_0}\int^{Y} e^{(y^2 - C_0)/(4\lambda\alpha_0)} dy.$$

Dawson function:

$$\int^x e^{y^2} dy \underset{x\to\infty}{\sim} \frac{1}{2x} e^{x^2} \Longrightarrow \frac{r_{\rm eff}}{\sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\rm eff}}} \underset{t\to\infty}{\sim} t.$$

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Conclusion

$u_0(x) = b_0 e^{-\frac{1}{2}\sum_{j=1}^d a_{0j}x_j^2}$, with $b_0, a_{0j} \in \mathbb{C}$, $\alpha_{0j} = \operatorname{Re} a_{0j} > 0$. Then

$$u(t,x) = b_0 \prod_{j=1}^d \frac{1}{\sqrt{r_j(t)}} \exp\left(i\phi_j(t) - \alpha_{0j}\frac{x_j^2}{2r_j^2(t)} + i\frac{\dot r_j(t)}{r_j(t)}\frac{x_j^2}{2}\right)$$

for some real-valued functions ϕ_j, r_j depending on t only.

$$r_j(t) = 2t\sqrt{\lambda\alpha_{0j}\ln t}\left(1+o(1)\right), \quad \dot r_j(t) = 2\sqrt{\lambda\alpha_{0j}\ln t}\left(1+o(1)\right).$$

$\rightsquigarrow$ Main (space dependent) oscillation:

$$\frac{\dot r_j(t)}{r_j(t)}\frac{x_j^2}{2} \underset{t\to\infty}{\sim} \frac{x_j^2}{2t} \underset{t\to\infty}{\sim} \frac{\dot\tau}{\tau}\frac{x_j^2}{2}.$$

$\rightsquigarrow$ Asymptotic variance:

$$\alpha_{0j}\frac{x_j^2}{2r_j^2(t)} \underset{t\to\infty}{\sim} \alpha_{0j}\frac{x_j^2}{8t^2\lambda\alpha_{0j}\ln t} = \frac{x_j^2}{8t^2\lambda\ln t} \underset{t\to\infty}{\sim} \frac{x_j^2}{2\tau(t)^2}.$$

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To be continued...

These remarks motivate the change of unknown function:

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \exp\left(i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}\right) \frac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}},$$

where $\gamma(x) = e^{-|x|^2/2}$.