Universal dynamics for the logarithmic Schrödinger equation
Lecture 1: general presentation and main results

Rémi Carles

CNRS & Univ Rennes

Based on a joint work with
Isabelle Gallagher (ENS Paris)

Heat equation

$$\partial_t u = \frac{1}{2}\Delta u, \quad x \in \mathbb{R}^d, \quad u_{|t=0} = u_0 \in L^1(\mathbb{R}^d).$$

Explicit solution : $u(t,x) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} u_0(y)dy.$

Large time description:

$$\hat{u}(t,\xi) = e^{-\frac{t}{2}|\xi|^2}\hat{u}_0(\xi) = \underbrace{e^{-\frac{t}{2}|\xi|^2}\hat{u}_0(0)}_{\text{order } t^{-d/(2p)} \text{ in } L^p} + \underbrace{e^{-\frac{t}{2}|\xi|^2}\left(\hat{u}_0(\xi) - \hat{u}_0(0)\right)}_{\mathcal{O}\left(|\xi|e^{-\frac{t}{2}|\xi|^2}\right)\text{: order } t^{-(d+1)/(2p)} \text{ in } L^p}$$

If $m := \int_{\mathbb{R}^d} u_0 \neq 0$,

$$u(t,x) \underset{t\to\infty}{\sim} \frac{m}{(2\pi t)^{d/2}} e^{-|x|^2/(2t)}.$$

$\rightsquigarrow$ Universal Gaussian profile.

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$\rightsquigarrow$ Universal oscillation, but the profile depends on the initial data.

Example (Explicit computation in the Gaussian case)

$$\mathrm{Re}\, z > 0 : \quad e^{i\frac{t}{2}\Delta}\left(e^{-z\frac{|x|^2}{2}}\right) = \frac{1}{(1+itz)^{d/2}} e^{-\frac{z}{1+itz}\frac{|x|^2}{2}}.$$

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$$u(t,x) = \frac{1}{(2i\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} u_0(y)dy = M_t D_t \mathcal{F} M_t u_0(x),$$

where the three operators,

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Nonlinear Schrödinger equation: power nonlinearity

For $\lambda \in \mathbb{R}$, $0 < \sigma < \frac{2}{(d-2)_+}$, consider:

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- If $\lambda > 0$: global existence ($u \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^d))$), and if $\sigma > 2/d$,

$$\exists u_+ \in H^1(\mathbb{R}^d), \quad \|u(t) - e^{i\frac{t}{2}\Delta} u_+\|_{H^1(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0.$$

  The (inverse of) the wave operator is not trivial: $u_0 \mapsto u_+$ is one-to-one.

- If $\lambda < 0$: finite time blow-up is possible when $\sigma \geqslant 2/d$,

$$\lim_{t\to T^*} \|\nabla u(t)\|_{L^2} = \infty.$$

  Existence of large stationary solutions: $u(t,x) = e^{i\omega t}\psi(x)$.

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  Existence of large stationary solutions: $u(t,x) = e^{i\omega t}\psi(x)$.

Logarithmic nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0.$$

$\rightsquigarrow$ Initial physical motivation (≈1976-1985): nonlinear wave mechanics, open quantum systems, nuclear physics.
More recent interest (since 2000): quantum optics, transport and diffusion phenomena, effective quantum gravity, superfluidity and BEC.
$\rightsquigarrow$ Formal conservations:

- Mass: $M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^d)}$.
- Momentum: $J(u(t)) := \mathrm{Im} \int_{\mathbb{R}^d} \overline{u}(t,x)\nabla u(t,x)dx$.
- Energy (Hamiltonian):
$$E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \ln|u(t,x)|^2 dx.$$

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Why is the Cauchy problem delicate?

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0.$$

$\rightsquigarrow$ The nonlinearity is not locally Lipschitz continuous: the usual fixed point argument cannot be repeated.

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A priori estimates: case $\lambda < 0$

Suppose $\lambda < 0$: positive part of the energy,

$$0 \leqslant E_+(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{|u|<1} |u(t,x)|^2 \ln |u(t,x)|^2 dx$$

$$\leqslant E(u_0) \underbrace{-\lambda}_{+|\lambda|} \int_{|u|>1} |u(t,x)|^2 \ln |u(t,x)|^2 dx.$$

Since the logarithm grows slowly,

$$\int_{|u|>1} |u(t,x)|^2 \ln |u(t,x)|^2 dx \leqslant C_\varepsilon \int_{|u|>1} |u(t,x)|^{2+\varepsilon} dx$$

$$\lesssim \|u(t)\|^{2+\varepsilon-\varepsilon d/2}_{L^2(\mathbb{R}^d)} \|\nabla u(t)\|^{\varepsilon d/2}_{L^2(\mathbb{R}^d)},$$

provided that $\varepsilon < 2d/(d-2)_+$. Using the conservation of mass,

$$E_+(u(t)) \leqslant E(u_0) + C_\varepsilon E_+\left(u(t)\right)^{\varepsilon d/4}.$$

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$$E_+(u(t)) \leqslant E(u_0) + C_\varepsilon E_+\left(u(t)\right)^{\varepsilon d/4}.$$

Picking $\varepsilon < 4/d$ yields $E_+\left(u(t)\right) \in L^\infty(\mathbb{R})$, hence

$$u \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^d)), \quad |u|^2 \ln |u|^2 \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^d)).$$

$\rightsquigarrow$ Nice a priori estimate in the case $\lambda < 0$, in

$$W := \left\{ u \in H^1(\mathbb{R}^d)\,,\ x \mapsto |u(x)|^2 \ln |u(x)|^2 \in L^1(\mathbb{R}^d) \right\}.$$

$\rightsquigarrow$ Mathematical study:

$$W := \left\{ u \in H^1(\mathbb{R}^d) \, , \, x \mapsto |u(x)|^2 \ln |u(x)|^2 \in L^1(\mathbb{R}^d) \right\} .$$

Theorem (Th. Cazenave & A. Haraux '80)

$\lambda < 0$, $u_0 \in W$: *unique, global solution* $u \in C(\mathbb{R}; W)$. *The mass* $M(u)$ *and the energy* $E(u)$ *are independent of time.*

- Proof by compactness arguments, using a regularization of the nonlinearity.
- Alternative proof by Masayuki Hayashi, proving the strong convergence of a sequence of approximate solutions.

A priori estimates: case $\lambda > 0$

If $\lambda > 0$,

$$0 \leqslant E_+(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{|u|>1} |u(t,x)|^2 \ln |u(t,x)|^2 dx$$
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Since the logarithm grows slowly,

$$\int_{|u|<1} |u(t,x)|^2 \ln \frac{1}{|u(t,x)|^2} dx \leqslant C_\varepsilon \int_{|u|<1} |u(t,x)|^{2-\varepsilon} dx,$$

for $\varepsilon > 0$ arbitrarily small. Analogue of Gagliardo–Nirenberg inequality,

$$\int_{\mathbb{R}^d} |u|^{2-\varepsilon} \lesssim \|u\|_{L^2}^{2-\varepsilon-\frac{d\varepsilon}{2\alpha}} \||x|^\alpha u\|_{L^2}^{\frac{d\varepsilon}{2\alpha}}, \quad 0 < \varepsilon < \frac{4\alpha}{d+2\alpha}.$$

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Cauchy problem: case $\lambda > 0$

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0.$$

$$\mathcal{F}(H^\alpha) = \left\{ u \in L^2(\mathbb{R}^d)\,,\ x \mapsto \langle x\rangle^\alpha u(x) \in L^2(\mathbb{R}^d) \right\},$$

Theorem (with I. Gallagher)

$\lambda > 0$, $u_0 \in \mathcal{F}(H^\alpha) \cap H^1(\mathbb{R}^d)$ with $0 < \alpha \leqslant 1$.
There exists a unique, global solution $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}; \mathcal{F}(H^\alpha) \cap H^1)$.
Mass $M(u)$ and energy $E(u)$ are independent of time.
If in addition $u_0 \in H^2(\mathbb{R}^d)$, then $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^2)$.

Remark

$\mathcal{F}(H^\alpha) \cap H^1 \subsetneq W$. Issue $= \{|u| < 1\}$.

Dynamics?

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0.$$

Whether $\lambda < 0$ or $\lambda > 0$, we now have the existence of a unique global solution.

What can we say about its large time behavior?

Usual NLS, $i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{2\sigma} u$.

- If $\lambda > 0$ (defocusing case): scattering if $\sigma > 1/d$, long range scattering for small data if $0 < \sigma \leqslant 1/d$. The scattering operator is bijective from a neighborhood of 0 to itself (non-trivial dynamics).
- If $\lambda < 0$ (focusing case): (long range) scattering for small data, but blow up may happen for large data when $\sigma \geqslant 2/d$.

The logarithmic Schrödinger equation has completely different features.

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Galilean invariance

Like the linear Schrödinger equation, or the “standard” NLS, logNLS enjoys translation invariance, and Galilean invariance:

If $u(t,x)$ solves

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u, \quad \lambda \in \mathbb{R},$$

then so does

$$u_v(t,x) = u(t,x-vt)e^{iv\cdot x - i|v|^2 t/2},$$

for any $v \in \mathbb{R}^d$.

Forget all about scaling

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u.$$

For $k > 0$, ku solves

$$i\partial_t(ku) + \frac{1}{2}\Delta(ku) = \lambda \ln\left(|u|^2\right) ku = \lambda \ln\left(|ku|^2\right) ku - \lambda\left(\ln k^2\right) ku.$$

Gauge transform: $u_k := (ku)e^{2it\lambda \ln k}$ solves the same equation as $u (= u_1)$.

$\rightsquigarrow$ A scaling factor does not change the dynamics.

$\rightsquigarrow$ Typical feature of linear equations.

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For instance:

$$\frac{du_k}{dk} = (1 + 2it\lambda)\, u_1 e^{2it\lambda \ln k}, \quad \frac{d^2 u_k}{dk^2} = \frac{2it\lambda}{k}(1 + 2it\lambda)\, u_1 e^{2it\lambda \ln k}.$$

For any $t \neq 0$, $u_0 \mapsto u(t)$ is not C^1 at 0 (even from $H^{+\infty}$ to $H^{-\infty}$).

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Logarithmic NLS: "focusing" case

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0.$$

Lemma (Th. Cazenave '83)

Let $\lambda < 0$ *and* $k < \infty$ *such that*

$$L_k := \{u \in W,\ \|u\|_{L^2(\mathbb{R}^d)} = 1,\ E(u) \leqslant k\} \neq \emptyset\,.$$

Then $\displaystyle\inf_{\substack{u\in L_k\\ 1\leqslant p\leqslant\infty}} \|u\|_{L^p(\mathbb{R})} > 0.$

$\rightsquigarrow$ No solution is dispersive in the case $\lambda < 0$.

$\rightsquigarrow$ standing wave: $\lambda < 0$, $\exp(-2i\lambda\omega t + \omega + d/2 + \lambda|x|^2)$ solution, for any period $\omega \in \mathbb{R}$.

$\rightsquigarrow$ Orbital stability of the Gausson ($\omega = 0$) addressed in several papers: Th. Cazenave '83, A. Ardila '16.

General dynamics, case $\lambda < 0$

For general initial data, the dynamics is not known is general:

- The Gausson is orbitally stable, but not stable in the strong sense of Lyapunov (counterexamples, even in the radial case).
- The evolution of any initial Gaussian is known (second lecture).
- For instance, what can we say about the evolution of the initial sum of two Gaussians??

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Gaussian data: the case $\lambda > 0$

Theorem

$\lambda > 0$, $u_0(x) = b_0 e^{-\frac{1}{2}\sum_{j=1}^d a_{0j}x_j^2}$, *with* $b_0, a_{0j} \in \mathbb{C}$, $\alpha_{0j} = \operatorname{Re} a_{0j} > 0$. *Then*

$$u(t,x) = b_0 \prod_{j=1}^d \frac{1}{\sqrt{r_j(t)}} \exp\left(i\phi_j(t) - \alpha_{0j}\frac{x_j^2}{2r_j^2(t)} + i\frac{\dot r_j(t)}{r_j(t)}\frac{x_j^2}{2}\right)$$

for some real-valued functions ϕ_j, r_j depending on t only.

$$r_j(t) = 2t\sqrt{\lambda\alpha_{0j}\ln t}\left(1+o(1)\right), \quad \dot r_j(t) = 2\sqrt{\lambda\alpha_{0j}\ln t}\left(1+o(1)\right).$$

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \underset{t\to\infty}{\sim} \frac{1}{\left(t\sqrt{\ln t}\right)^{d/2}} \frac{\|u_0\|_{L^2}}{\left(2\lambda\sqrt{2\pi}\right)^{d/2}},$$

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} 2\lambda d \|u_0\|^2_{L^2(\mathbb{R}^d)} \ln t.$$

Three new features

- Dispersion: usual $t^{-d/2}$ rate becomes $(t\sqrt{\ln t})^{-d/2}$. (Link with Strichartz estimates?)
- Unboundedness in H^1: $\|\nabla u(t)\|_{L^2} \approx \sqrt{\ln t}$.
- *Universal profile:*

$$(2t\sqrt{\lambda \ln t})^{d/2} \left| u\left(t, x \times 2t\sqrt{\lambda \ln t}\right) \right| \underset{t\to\infty}{\longrightarrow} \frac{\|u_0\|_{L^2}}{\pi^{d/4}} e^{-|x|^2/2},$$

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$$(2t\sqrt{\lambda \ln t})^{d/2} \left| u\left(t, x \times 2t\sqrt{\lambda \ln t}\right)\right| \underset{t\to\infty}{\longrightarrow} \frac{\|u_0\|_{L^2}}{\pi^{d/4}} e^{-|x|^2/2},$$

regardless of the initial variances.

$\rightsquigarrow$ Miraculous explicit computations, precious guide for the general case.

Three new features

- Dispersion: usual $t^{-d/2}$ rate becomes $(t\sqrt{\ln t})^{-d/2}$. (Link with Strichartz estimates?)
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Universal dispersion

Lemma

Let $\lambda > 0$. *Consider the ODE:*

$$\ddot{\tau} = \frac{2\lambda}{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0.$$

It has a unique solution $\tau \in C^2(0,\infty)$, *and, as* $t \to \infty$,

$$\tau(t) = 2t\sqrt{\lambda \ln t}\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big), \quad \dot{\tau}(t) = 2\sqrt{\lambda \ln t}\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big),$$

where $\ell(t) := \dfrac{\ln \ln t}{\ln t}$.

Remark

The large time behavior is independent of $\tau(0) > 0$ and $\dot{\tau}(0) \in \mathbb{R}$.

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Theorem

Let $u_0 \in H^1 \cap \mathcal{F}(H^1)$, *and* $\gamma(y) = e^{-|y|^2/2}$. *Define*

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \exp\left(i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}\right) \frac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}}.$$

$$\int_{\mathbb{R}^d} \left(1 + |y|^2 + \left|\ln|v(t,y)|^2\right|\right) |v(t,y)|^2 dy + \frac{1}{\tau(t)^2} \|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)} \leqslant C.$$

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) dy.$$

$$|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \textit{weakly in } L^1(\mathbb{R}^d).$$

Formally, $u(t,x) \underset{t\to\infty}{\sim} \dfrac{1}{(2t)^{d/2}(\lambda \ln t)^{d/4}} e^{-|x|^2/(4\lambda t^2 \ln t)} e^{i|x|^2/2t} \dfrac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}}$.

Positive Sobolev norms

Corollary

Let $u_0 \in H^1 \cap \mathcal{F}(H^1) \setminus \{0\}$, *and* $0 < s \leqslant 1$. *As* $t \to \infty$,

$$(\ln t)^{s/2} \lesssim \|u(t)\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim (\ln t)^{s/2},$$

where $\dot{H}^s(\mathbb{R}^d)$ *denotes the standard homogeneous Sobolev space.*

Proof in the case $s = 1$.

$$\nabla u(t,x) = \frac{1}{\tau(t)^{d/2}} \nabla_x \left(v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}} \right)$$

$$= \underbrace{\frac{1}{\tau(t)} \frac{1}{\tau(t)^{d/2}} \nabla_y v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}}}_{\|\cdot\|_{L^2} = \frac{1}{\tau}\|\nabla v\|_{L^2} = \mathcal{O}(1).} + \underbrace{i\dot\tau \frac{1}{\tau(t)^{d/2}} \frac{x}{\tau} v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}}}_{\|\cdot\|_{L^2} = \dot\tau\|yv\|_{L^2} \sim \dot\tau\|y\gamma\|_{L^2} \approx \sqrt{\ln t}}.$$

□

The Gaussian case

Corollary

Suppose $u_0(x) = b_0 e^{-\frac{1}{2}\sum_{j=1}^{d} a_{0j}x_j^2}$, *with* $b_0, a_{0j} \in \mathbb{C}$, $\alpha_{0j} = \operatorname{Re} a_{0j} > 0$. *Then, with* v *as in the main theorem, the relative entropy of* $|v|^2$ *goes to zero for large time:*

$$\int_{\mathbb{R}^d} |v(t,y)|^2 \ln \left| \frac{v(t,y)}{\gamma(y)} \right|^2 dy \underset{t\to\infty}{\longrightarrow} 0\,,$$

and the convergence of $|v|^2$ *to* γ^2 *is strong in* L^1*:*

$$\left\| |v(t,\cdot)|^2 - \gamma^2 \right\|_{L^1(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0.$$

Stability of this regime

If we consider a defocusing, energy-subcritical, power-like perturbation,

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u + \mu |u|^{2\sigma} u\,, \quad u_{|t=0} = u_0,$$

with $\lambda > 0$, $\mu > 0$ and $0 < \sigma < \frac{2}{(d-2)_+}$, then most of the previous results remain valid:

- Existence of a global solution in $\Sigma = H^1 \cap \mathcal{F}(H^1)$, with conservation of mass, momentum and energy (uniqueness is not always known).
- Main Theorem: with the same change of unknown function, convergence of the momenta, $|v(t,\cdot)|^2 \rightharpoonup \gamma^2$, and growth of the Sobolev norms of u.

Remark

In this case, the evolution of Gaussian data is not explicit.

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Outline of the proof of the main theorem

- Gaussian case: reduction to ODEs. Appearance of τ.
- This suggests the new unknown function v, which solves

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2, \quad v_{|t=0} = u_0.$$

- A priori estimates for v: directly from the above equation, and from the a priori estimates for u.
- Proof of $|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2$:
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Content of the other lectures

- Lecture 2: Cauchy problem and explicit Gaussian solutions.
- Lecture 3: proof of the main theorem, and corollaries.