Measurements, uncertainties and probabilistic inference/forecasting

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https://www.roma1.infn.it/~dagos/AQ2021/
(temporary web page)

Falsificationism and p-values

- arXiv:physics/0412148 [physics.data-an]
- arXiv:1112.3620 [physics.data-an]
- arXiv:1609.01668 [physics.data-an]

Testing one hypothesis

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- In practice:
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  → how is it done?

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Let's review the practice and what is behind it ⇒

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It seems OK – *'obvious'*! – but it is indeed naïve for several aspects.

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- Assume that a hypothesis is true;
- Derive 'all' logical consequences;
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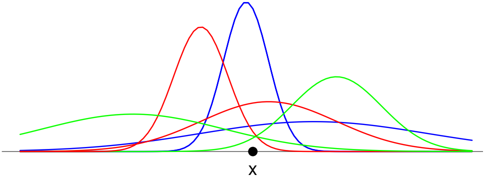
  $\Rightarrow$ Given any pair or parameters $\{\mu_i, \sigma_i\}$ (i.e. $\forall H_i$), all values of x from $-\infty$ to $+\infty$ are possible.

  $\Rightarrow$ Having observed any value **x**, none of H_i can be, strictly speaking, falsified.



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⇒ logically speaking, Popper's falsificationism
has to be considered ... falsified!

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[*I am particularly worried about claims concerning our health, or the status of the Planet, etc. ...*]

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B) ~~**if** $C_i \xrightarrow[\text{small probability}]{} E$, and **we observe** E~~ NO

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~~"most likely false"~~

But it is behind the rational behind the statistical hypothesis tests!

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In particular

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What else?

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(From the logical point of view the situation gets worsened:
$\rightarrow$ conclusions depend on events not actually observed!)

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→ the test variable ('θ') is absolutely arbitrary:

$$\theta = \theta(\mathrm{x})$$

$$\rightarrow f(\theta) \text{ [p.d.f]}$$

$$\text{Experiment:} \rightarrow \theta_{obs} = \theta(\mathrm{x}_{obs})$$

$$\text{p-value} = P(\theta \geq \theta_{obs}) \qquad \text{('one tail')}$$

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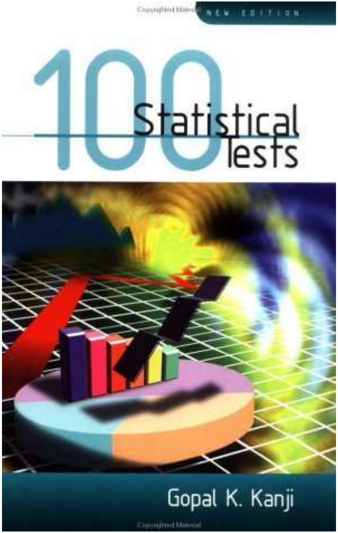

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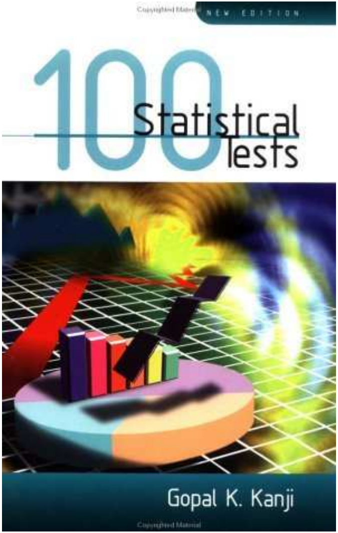

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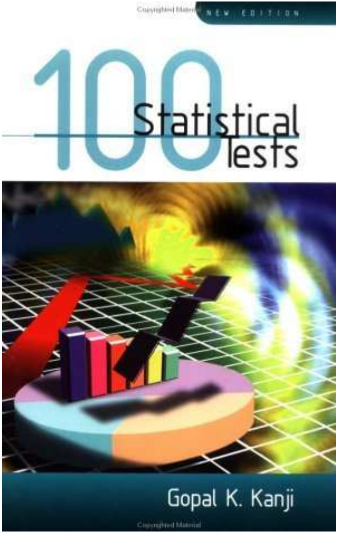

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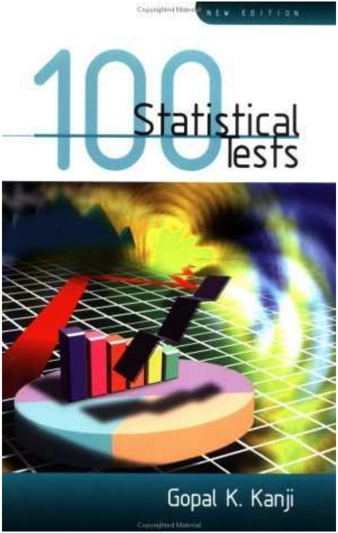

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→ *like if you go around until "someone agrees with you"*

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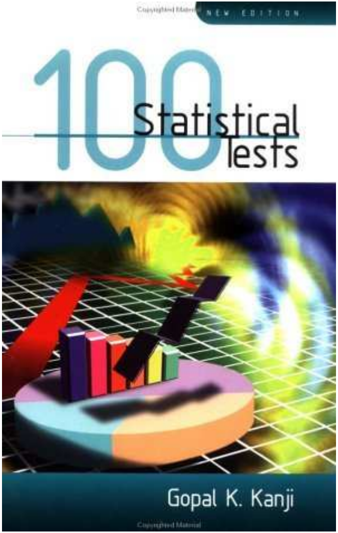

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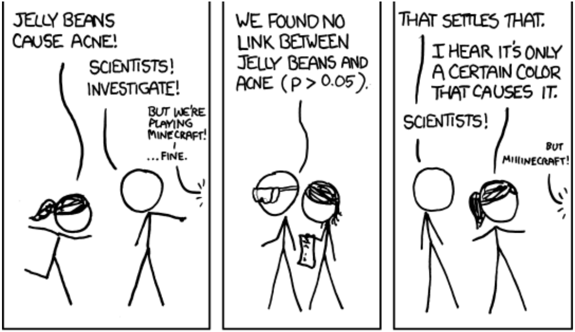
- personal **'golden rule'**:
"the more exotic is the name of the test, the less I believe the result", because I'm pretty sure that several 'normal' tests have been discarded in the meanwhile...

Or look around, searching for ‘significance’

If changing the test does not help, change hypotheses...



[http://xkcd.com/882/]

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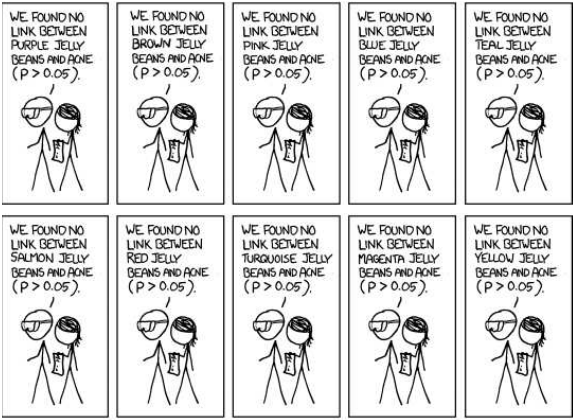

[http://xkcd.com/882/]

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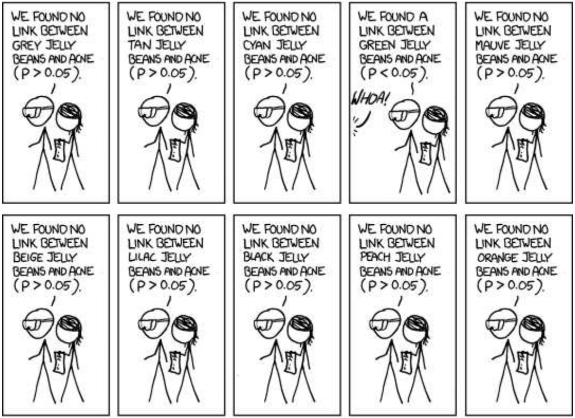

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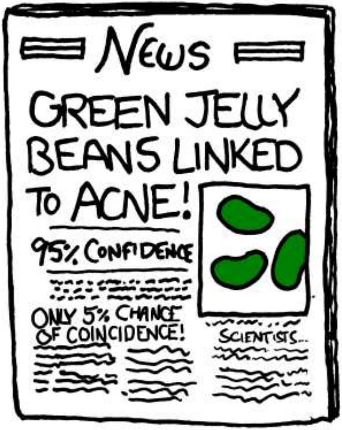

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P-hacking ("p-value hacking")

The 'science' of inventing significant results. . .

p-hacking, or cheating on a p-value

June 11, 2015

By arthur charpentier

Share

(This article was first published on **Freakonometrics » R-english**, and kindly contributed to R-bloggers)

Yesterday evening, I discovered some interesting slides on False-Positives, p-Hacking, Statistical Power, and Evidential Value, via @UCBITSS 's post on Twitter. More precisely, there was this slide on how cheating (because that's basically what it is) to get a 'good' model (by targeting the *p*-value)

1. Stop collecting data once $p<.05$
2. Analyze many measures, but report only those with $p<.05$.
3. Collect and analyze many conditions, but only report those with $p<.05$.
4. Use covariates to get $p<.05$.
5. Exclude participants to get $p<.05$.
6. Transform the data to get $p<.05$.

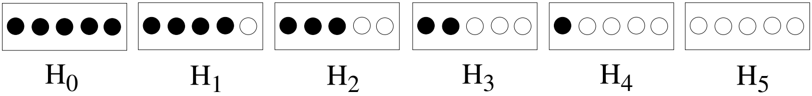
http://www.r-bloggers.com/p-hacking-or-cheating-on-a-p-value/

- Google for "p-hacking"

Continuing

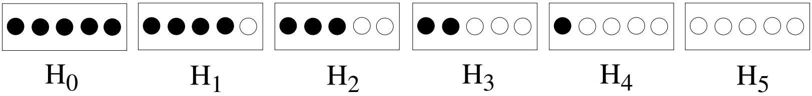
from last lecture

Which box? Which ball?



Let us take randomly one of the boxes.

Which box? Which ball?



Let us take randomly one of the boxes.

We are in a state of uncertainty concerning several *events*, the most important of which correspond to the following questions:

(a) Which box have we chosen, H_0, H_1, ..., H_5?

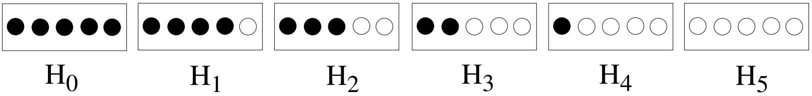
(b) If we extract randomly a ball from the chosen box, will we observe a white ($E_W \equiv E_1$) or black ($E_B \equiv E_2$) ball?

Our certainties:

$$\cup_{j=0}^{5} H_j = \Omega$$

$$\cup_{i=1}^{2} E_i = \Omega\,.$$

Which box? Which ball?



Let us take randomly one of the boxes.

- What happens after we have extracted one ball and looked its color?
  - Intuitively feel *how to* roughly *change* our opinion about
    - the possible cause
    - a future observation

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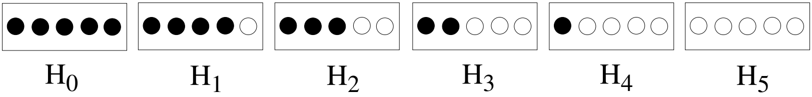

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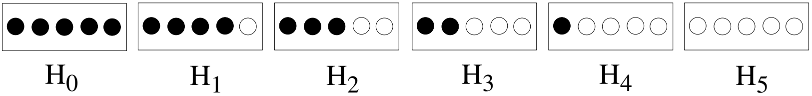

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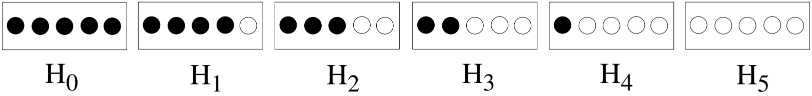

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Note: In general, we are uncertain about all the combinations of E_i and H_j:

$$E_1 \cap H_0,\ E_1 \cap H_1,\ \ldots,\ E_2 \cap H_5,$$

and these 12 constituents are not equiprobable.

Subjective nature of probability

"Since the knowledge may be different with different persons

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Probability depends on the status of information of the *subject* who evaluates it.

Probability is always conditional probability

"Thus whenever we speak loosely of 'the probability of an event', it is always to be understood: probability with regard to a certain given state of knowledge"

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$$P(E) \longrightarrow P(E \mid I_s(t))$$

where $I_s(t)$ is the information available to *subject* s at time t.

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Examples:

- tossing coins and dice;
- the three box problem.

What are we talking about?

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"Given the state of our knowledge about everything that could possible have any bearing on the coming true... the numerical probability P of this event is to be a real number by the indication of which we try in some cases to setup a quantitative measure of the strength of our conjecture or anticipation, founded on the said knowledge, that the event comes true"

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$\Rightarrow$ How much we believe something

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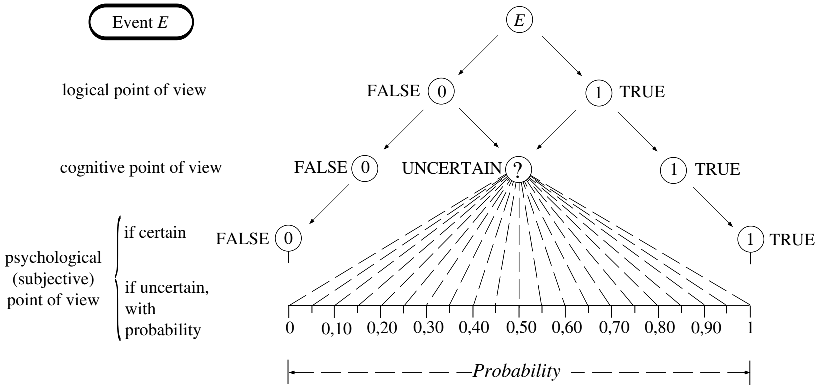
$\rightarrow$ 'Degree of belief' $\leftarrow$

False, True and probable



Beliefs and 'coherent' bets

Remarks:

- Subjective does not mean arbitrary!

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“His [Bouvard] calculations give him the mass of Saturn as 3,512th part of that of the sun. Applying my probabilistic formulae to these observations, I find that the odds are 11,000 to 1 that the error in this result is not a hundredth of its value.” (Laplace)

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Is a ‘conventional’ 95% C.L. lower/upper bound a 19 to 1 bet?

Standard textbook definitions

$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$p = \frac{\#\,\text{times the event has occurred}}{\#\,\text{independent trials under same conditions}}$$

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It is easy to check that 'scientific' definitions suffer of circularity

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$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equally possible cases}}$$

$$p = \frac{\#\,\text{times the event has occurred}}{\#\,\text{independent trials under same conditions}}$$

Note!: *"lorsque rien ne porte à croire que l'un de ces cas doit arriver plutot que les autres" (Laplace)*

Replacing 'equi-probable' by 'equi-possible' is just cheating students (as I did in my first lecture on the subject...).

Standard textbook definitions

It is easy to check that 'scientific' definitions suffer of circularity, plus other problems

$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$p = \lim_{n\rightarrow\infty} \frac{\#\,\text{times the event has occurred}}{\#\,\text{independent trials under same condition}}$$

Future $\Leftrightarrow$ Past (belief!)

$n \rightarrow \infty$: $\rightarrow$ *"usque tandem?"*

$\rightarrow$ *"in the long run we are all dead"*

$\rightarrow$ It limits the range of applications

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Future ⇔ Past (belief!)

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Future ⇔ Past: avoid the end of the *inductivist turkey*!

‘Definitions’ $\rightarrow$ evaluation rules

Very useful evaluation rules

$$A) \quad p \;=\; \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$B) \quad p \;=\; \frac{\#\,\text{times the event has occurred}}{\#\text{independent trials under same condition}}$$

If the implicit beliefs are well suited for each case of application.

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BUT they cannot define the concept of probability!

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- Rule B results from a theorem of Probability Theory (under well defined assumptions):
  ⇒ Laplace’s rule of succession (see later)

Mathematics of beliefs

The good news:

> *The basic laws of degrees of belief are the same we get from the inventory of favorable and possible cases, or from events occurred in the past.*

It can be proved that

> *the requirement of coherence leads to the famous 4 basic rules* $\Longrightarrow$

[Details skipped. . .]

Basic rules of probability

1. $0 \le P(A \mid I) \le 1$
2. $P(\Omega \mid I) = 1$
3. $P(A \cup B \mid I) = P(A \mid I) + P(B \mid I) \quad [\text{if } P(A \cap B \mid I) = \emptyset]$
4. $P(A \cap B \mid I) = P(A \mid B, I) \cdot P(B \mid I) = P(B \mid A, I) \cdot P(A \mid I)$

Remember that probability is always conditional probability!

I is the background condition (related to information 'I'_s')

→ usually implicit (we only care about 're-conditioning')

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Note: 4. does not define conditional probability.
(Probability is always conditional probability!)

Mathematics of beliefs

An even <u>better news</u>:

The fourth basic rule
can be fully exploited!

Mathematics of beliefs

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(Liberated by a curious ideology that forbids its use)

A simple, powerful formula

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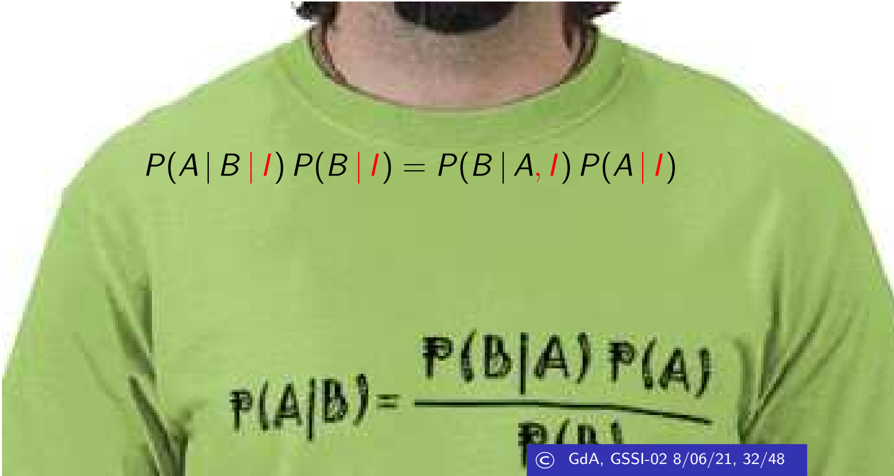
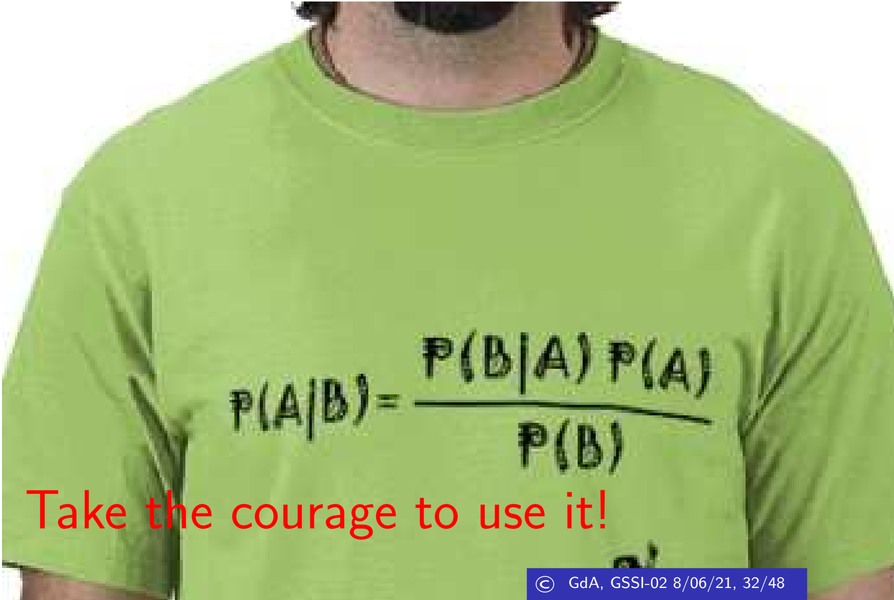


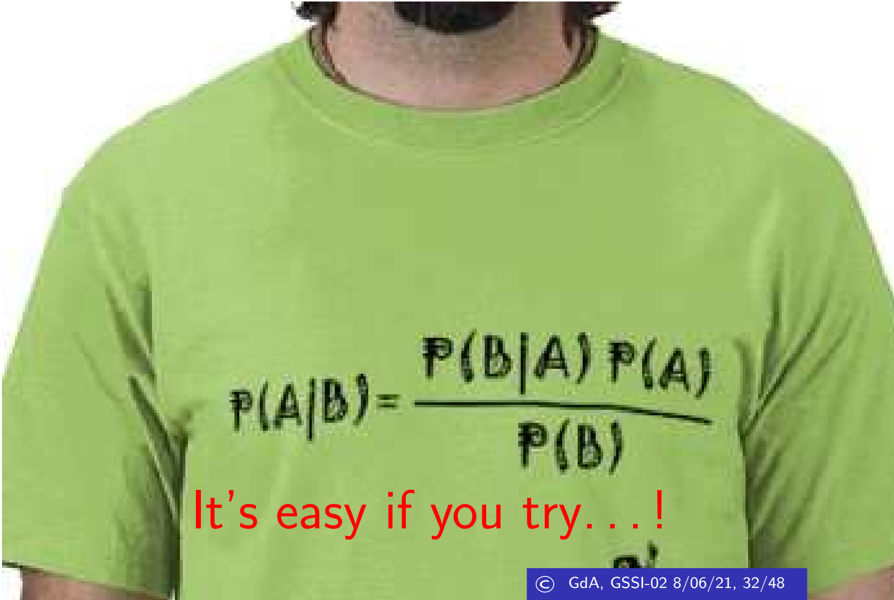
$$P(A \,|\, B \,|\, I)\, P(B \,|\, I) = P(B \,|\, A, I)\, P(A \,|\, I)$$

A simple, powerful formula



Take the courage to use it!

A simple, powerful formula



It's easy if you try. . . !

A simple, powerful formula

[Bayes Theorem]

A nice and powerful formula!

GdA and Allen Caldwell, Stellenbosch, South Africa, November 2013

[T-shirts kindly provided by Pangea Formazione]

Laplace's "Bayes Theorem"

"The greater the probability of an observed event given any one of a number of causes to which that event may be attributed, the greater the likelihood of that cause {given that event}.

$$P(C_i \,|\, E) \propto P(E \,|\, C_i)$$

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$$P(C_i \,|\, E) = \frac{P(E \,|\, C_i)\, P(C_i)}{P(E)}$$

(*Philosophical Essai on Probabilities*)

[In general $P(E) = \sum_j P(E \,|\, C_j)\, P(C_j)$ (*weighted average*, with weigths being the probabilities of the conditions) if C_j form a *complete class of hypotheses*]

Laplace's "Bayes Theorem"

$$P(C_i \,|\, E) \;=\; \frac{P(E \,|\, C_i)\, P(C_i)}{P(E)} = \frac{P(E \,|\, C_i)\, P(C_i)}{\sum_j P(E \,|\, C_j)\, P(C_j)}$$

"This is the fundamental principle (*) of that branch of the analysis of chance that consists of reasoning a posteriori from events to causes"

(*) In his "Philosophical essay" Laplace calls 'principles' the 'fundamental rules'.

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Most convenient way to remember Bayes theorem

Laplace's teaching

$$\frac{P(H_0 \,|\, \text{data})}{P(H_1 \,|\, \text{data})} \;=\; \frac{P(\text{data} \,|\, H_0)}{P(\text{data} \,|\, H_1)} \times \frac{P(H_0)}{P(H_1)}$$

- We should possibly use the data, rather then the test variables 'θ' (χ^2 etc);

  [although in some case 'sufficient summaries' do exist]

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  $\Rightarrow$ falsification (the 'serious' one) is a corollary of the theorem, rather than a principle.
- There is no conceptual problem with the fact that $P(\text{data} \,|\, H_1) \rightarrow 0$ (e.g. 10^{-37}), provided the ratio $P(\text{data} \,|\, H_0)/P(\text{data} \,|\, H_1)$ is not undefined.

Bayes factor ('likelihood ratio')

$$\frac{P(H_0\,|\,\text{data})}{P(H_1\,|\,\text{data})} \quad = \quad \frac{P(\text{data}\,|\,H_0)}{P(\text{data}\,|\,H_1)} \times \frac{P(H_0)}{P(H_1)}$$

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$$\text{Prob. ratio}\big|_{posterior} \;=\; \text{Bayes factor} \times \text{Prob. ratio}\big|_{prior}$$

(*prior*/*posterior* w.r.t. data)

Bayes factor ('likelihood ratio')

$$\frac{P(H_0\,|\,\text{data})}{P(H_1\,|\,\text{data})} \;=\; \frac{P(\text{data}\,|\,H_0)}{P(\text{data}\,|\,H_1)} \times \frac{P(H_0)}{P(H_1)}$$

$$\text{Prob. ratio}\big|_{posterior} \;=\; \text{Bayes factor} \times \text{Prob. ratio}\big|_{prior}$$

(*prior*/*posterior* w.r.t. data)

If H_0 and H_1 are 'complementary', that is $H_1 = \overline{H}_0$, then

posterior odds = Bayes factor × prior odds

Application to the Aids test problem

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And, needless to say, try to think to Covid-19 test issues:

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- dependence on the fact that the test performances unavoidably some degree of uncertanty.

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- Certainly! I agree!
  As it *usually* work overtakes in curve
  on remote mountain road!
- But now we are also able to explain the reason.

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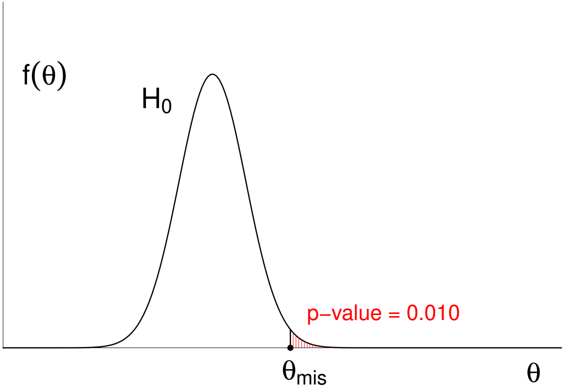

Why should the observation of θ_{mis} should diminish our confidence on H_0?

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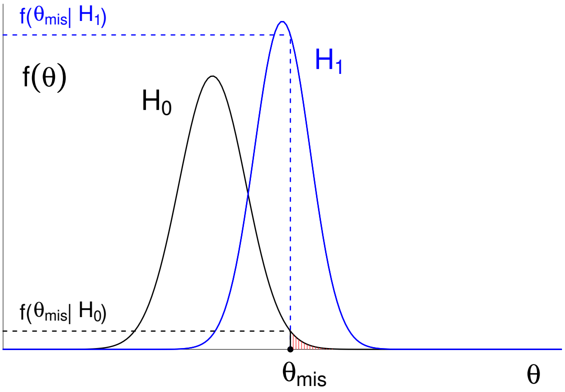

Because *often* we give *some chance* to a possible alternative hypothesis H_1, even if we are not able to exactly formulate it.

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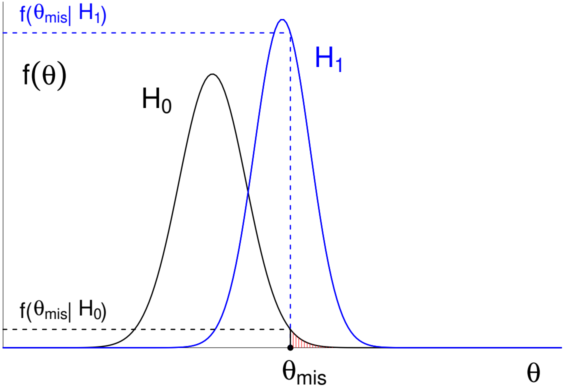

Indeed, what really matters <u>is not</u> the area to the right of θ_{mis}.
What matters is the ratio of $f(\theta_{mis}\,|\,H_1)$ to $f(\theta_{mis}\,|\,H_0)$!
$\Rightarrow$ to a 'small' area it corresponds a 'small' $f(\theta_{mis}\,|\,H_0)$.

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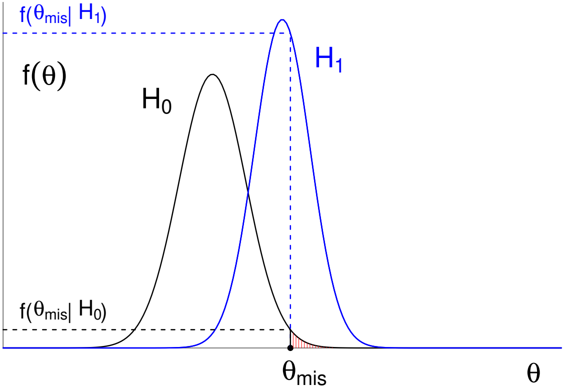

But is the alternative hypothesis H_1 is unconceivable, or hardly believable, the 'smallness' of the area is irrelevant

Telling it with Gauss' words

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Arguments used to derive Gaussian distribution

- $f(\mu\,|\,\{x\}) \propto f(\{x\}\,|\,\mu)\cdot f_0(\mu)$
- $f_0(\mu)$ 'flat' (all values a priory equally possible)
- posterior maximized at $\mu = \overline{x}$

The Gauss' Bayes Factor

It might be curious to learn that Gauss had proved, **with emphasis**, the rule to update the ratio of probabilities of complementary hypotheses, in the light of an observed event which could be due to either of them.

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⇒ `arXiv:2003.10878 [math.HO]`

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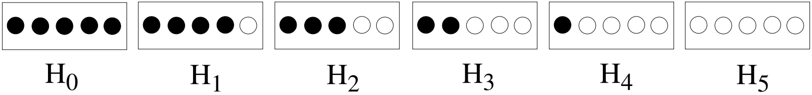
$\Rightarrow$ `arXiv:2003.10878 [math.HO]`

(And, by the way, Enrico Fermi derived analysis tools based on *his* Bayes Theorem...

$\Rightarrow$ `arXiv:physics/0509080 [physics.hist-ph]`)

Application to the six box problem



Remind:

- $E_1 =$ White
- $E_2 =$ Black

Collecting the pieces of information we need

Our tool:

$$P(H_j \mid E_i, I) = \frac{P(E_i \mid H_j, I)}{P(E_i \mid I)} P(H_j \mid I)$$

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Our prior belief about H_j

Collecting the pieces of information we need

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Probability of E_i under a well defined hypothesis H_j
It corresponds to the 'response of the apparatus' in measurements.
→ likelihood (traditional, rather confusing name!)

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$\rightarrow$ How much we are confident that E_i will occur.

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Probability of E_i taking account all possible H_j
→ How much we are confident that E_i will occur.
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'decomposition law': $P(E_i \mid I) = \sum_j P(E_i \mid H_j, I) \cdot P(H_j \mid I)$
($\rightarrow$ Easy to check that it gives $P(E_i \mid I) = 1/2$ in our case).

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$\longrightarrow$ Let's play with our toy

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Now that we have set up our formalism, let's play a little

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- some simulations
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Let's play!

- Hugin Expert (Lite – demo version);
- R scripts

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NO!

- Don't even think: frequentists refuse to assign probabilities to hypotheses (in general), to causes, to true values, etc.
  (And you have seen the results. . .)

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Simple case (no reporter/composition/etc. complications)

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- Update probabilities of next extraction: *prediction*:

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No real distinction between inference and prediction
(We shall see it later in the case of *continuous distributions*)

The End