1. Simulation of the Wiener process and its discretization

$$dX(t) = f(t, X(t))\,dt + g(t, X(t))\,dW(t), \quad t \in [0, T]$$

$\dot{y} = f(y)$ ODE $\overset{\text{discretization}}{\rightsquigarrow}$ 1. $[t_0, T] \to \mathcal{I}_h$

in $t \in [t_0, T]$ 2. $y(t) \to$ num. solution

$$\mathcal{I}_h = \{ t_n = t_0 + nh,\ n = 0, \dots, N,\ N = \frac{T - t_0}{h} \}, \quad h > 0$$

$$\begin{array}{ccccc} t_0, & t_1, & t_2, & \dots, & t_N \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ y_0 & y_1 & y_2 & \dots & y_N \end{array}$$

$$y_i \approx y(t_i), \quad i = 0, 1, \dots, N$$

The numerical solution is then the set

$$\{ y_0, y_1, y_2, \dots, y_N \}$$

represented as a matrix:

$$Y = [y_0\ y_1\ \dots\ y_N] \in \mathbb{R}^{d \times (N+1)}$$

$$y_0 \in \mathbb{R}^d$$

3. $\dot{y} = f(y) \rightsquigarrow$ difference equation

$$\begin{array}{ccc} \dot{y} = f(y) & \rightsquigarrow & \text{difference equation} \\ \uparrow & & \uparrow \\ y(t) & & \{y_0, y_1, \dots, y_N\} \end{array}$$

For SDEs, you need a fourth level of discretization, required for the Itō integral.

$$\int_0^t g(s, X(s))\,dW(s) \approx \sum_{i=1}^{n} g(t_i, X(\underset{\tau_i}{t_i})) \cdot \boxed{(W(\underset{\tau_{i+1}}{t_{i+1}}) - W(\underset{\tau_i}{t_i}))}$$

where τ_i, $i = 1, \dots, n$, are the Itō quadrature points.

Brief remainder

The standard Wiener process (or standard Brownian motion) in $[0, T]$ is the stochastic process $\{W(t), t \in [0, T]\}$ satisfying the following properties:

1) $W(0) = 0$ with probability 1;
2) $\forall\ 0 \le s < t \le T$

$$\underbrace{W(t) - W(s)}_{\text{Wiener increment}} \sim \sqrt{t - s}\ \underbrace{\mathcal{N}(0,1)}_{\text{normal standard random variable}}$$

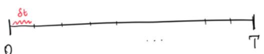
3) $\forall\ 0 \le s < t < u < v \le T$, the increments

$$W(t) - W(s), \quad W(v) - W(u)$$

are independent.

AIM provide a discretized Wiener increment, i.e., we specify values of $W(t)$ in a selection of discrete points in $[0, T]$.

As usual,



Let us define $\delta t = \frac{T}{N}$

$t_0 \equiv 0, t_1, t_2, \dots, t_N$, where

$$t_j = j\,\delta t, \quad j = 0, 1, \dots, N.$$

Notation: $W_j := W(t_j)$.

Goal: $\{ W_0, W_1, \dots, W_N \}$.

Algorithm:

- $W_0 = 0$ (with probability 1)
- $W_1 = W_0 + dW_1 \sim \sqrt{\delta t} \cdot \mathcal{N}(0,1)$

$\vdots$

- $W_j = W_{j-1} + dW_j \sim \sqrt{\delta t} \cdot \mathcal{N}(0,1)$

$$j = 1, \dots, N$$

In Matlab the simulation of $\mathcal{N}(0,1)$ occurs by employing the built-in function **randn**

N increments (due to their independence) computable altogether:

randn(1, N)

$$W_0 = 0$$
$$W_1 = W_0 + dW_1 = dW_1$$
$$W_2 = W_1 + dW_2 = dW_1 + dW_2$$
$$\vdots$$
$$W_K = \sum_{i=1}^{K} dW_K$$

$\leftarrow$ cumulated summation of the vector of the increments

cumsum

Example

```
a = 1:5;
b = cumsum(a)
  = [1 3 6 10 15]
```

Exercise

$u(W(t)) = \exp\left(t + \frac{1}{2} W(t)\right)$

over $M = 1000$ different trajectories

N : number of points for the computation of the Wiener increments

randn(M,N)

$$W_j^i = W_{j-1}^i + dW_j^i$$

i → i-th path

j → j-th point

W = cumsum(dW,2)

example $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

$B = \text{cumsum}(A,2) = \begin{bmatrix} 1 & 4 & 9 \\ 2 & 6 & 12 \end{bmatrix}$.

$$u(W(t)) = \exp\Big(\underbrace{t}_{\in \mathbb{R}^N} + \frac{1}{2} \underbrace{W(t)}_{\in \mathbb{R}^{M\times N}}\Big)$$

$$\begin{bmatrix} t^T \\ t^T \\ \vdots \\ t^T \end{bmatrix} \in \mathbb{R}^{M\times N} \rightsquigarrow \text{repmat}(t,[M,1])$$

Exact expectation:

$$\mathbb{E}\left[u(W(t))\right] = \exp\left(\frac{9}{8} t\right),$$

vs

computation of an estimate expectation.

MONTE CARLO SIMULATIONS

* MULTILEVEL MONTE CARLO SIMULATIONS
  Giles

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2. Simulation of Itō integrals

$$\int_0^T h(t)\,dW(t) = \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} h(t_j) \cdot \left[W(t_{j+1}) - W(t_j)\right]$$

$$\int_0^T h(t)\,dW(t) \approx \sum_{j=0}^{N-1} h(t_j) \cdot \left[W(t_{j+1}) - W(t_j)\right]$$

for large enough values of N.

Exercise: $\displaystyle\int_0^T W(t)\,dW(t) = \frac{1}{2}\left(W^2(T) - T\right)$

(Kloeden-Platen, 1992)

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3. Itō-Taylor expansions

ODEs

- Taylor series arguments
  (e.g., Euler method)
- numerical quadrature
  (e.g., Milne-Simpson)
- Gaussian quadrature $\nearrow y_0 \to y_1$
  (e.g., Runge-Kutta)
- polynomial interpolation arguments
  (e.g., Adams methods)
- Lie integrators

~ o ~

Taylor series methods for SDEs?

Yes, but ... Itō calculus does not obey to the classical chain rule

Stratonovich calculus does

Itō lemma

let $X(t)$, $t \in [0,T]$, be the solution of the Itō problem

Itō process

$$X(t) = X(0) + \int_0^t f(s,X(s))\,ds + \int_0^t g(s,X(s))\,dW(s).$$

We focus on the case $d = m = 1$.

Moreover, let us consider a function

$$V \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R},\mathbb{R}).$$

Then, $V(t,X(t))$ satisfies the following SDE:

Itō formula

$$V(t,X(t)) = V(0,X(0)) + \int_0^t \Big[V_t(s,X(s)) + V_x(s,X(s))\,f(s,X(s)) + \frac{1}{2} V_{xx}(s,X(s))\,g^2(s,X(s))\Big]ds + \int_0^t V_x(s,X(s))\,g(s,X(s))\,dW(s).$$

Itō equation $\displaystyle X(t) = X(0) + \int_0^t f(s,X(s))\,ds + \int_0^t g(s,X(s))\,dW(s)$

let us apply Itō formula to compute $f(s,X(s))$ and $g(s,X(s))$:

Itō lemma $V \equiv f$

$$f(s,X(s)) = f(0,X(0)) + \int_0^t \Big[f_t(u,X(u)) + f_x(u,X(u))\,f(u,X(u)) + \frac{1}{2} f_{xx}(u,X(u))\,g^2(u,X(u))\Big]du + \int_0^t f_x(u,X(u))\,g(u,X(u))\,dW(u).$$

Itō lemma $V \equiv g$

$$g(s,X(s)) = g(0,X(0)) + \int_0^t \Big[g_t(u,X(u)) + g_x(u,X(u))\,f(u,X(u)) + \frac{1}{2} g_{xx}(u,X(u))\,g^2(u,X(u))\Big]du + \int_0^t g_x(u,X(u))\,g(u,X(u))\,dW(u)$$

Recursively applying the Itō formula in above expressions for the drift and the diffusion of an Itō SDEs provides their so-called Itō-Taylor expansion.

Let us approximate $f(t,X(t))$ and $g(t,X(t))$ by their first term in the Itō-Taylor expansion:

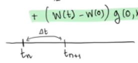
$$f(t,X(t)) \approx f(0,X(0))$$

$$g(t,X(t)) \approx g(0,X(0))$$

$$X(t) = X(0) + \int_0^t f(s,X(s))\,ds + \int_0^t g(s,X(s))\,dW(s)$$

$$\approx X(0) + \int_0^t f(0,X(0))\,ds + \int_0^t g(0,X(0))\,dW(s)$$

$$= X(0) + t \cdot f(0,X(0)) + \big(W(t) - W(0)\big)\,g(0,X(0))$$



$$X(t_{n+1}) \approx X(t_n) + \Delta t \cdot f(t_n, X(t_n)) + \big(W(t_{n+1}) - W(t_n)\big) \cdot g(t_n, X(t_n))$$

$X_n \approx X(t_n)$

$$\boxed{X_{n+1} = X_n + \Delta t\, f(t_n, X_n) + \big(W(t_{n+1}) - W(t_n)\big) \cdot g(t_n, X_n)}$$

EULER-MARUYAMA

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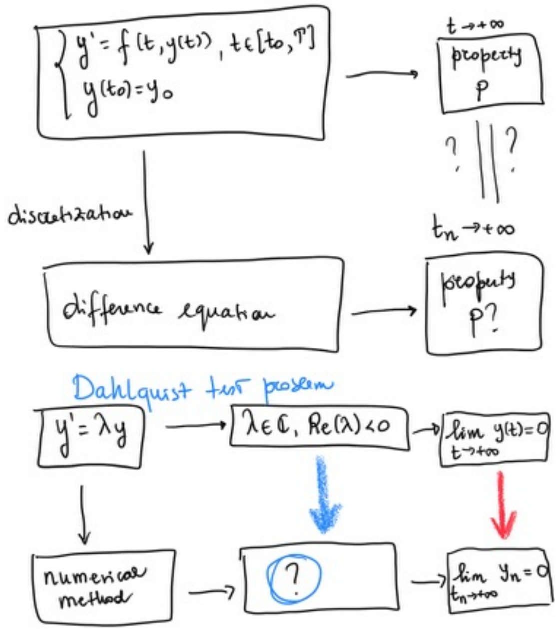
Stability

Lambert (ODEs)



SDEs

Stochastic Dahlquist test problem:

$$dX(t) = \lambda X(t)\,dt + \mu X(t)\,dW(t), \quad \lambda, \mu \in \mathbb{C}$$

Mean-square stability

$$\lim_{t \to +\infty} \mathbb{E}\left[|X(t)|^2\right] = 0 \iff Re(\lambda) + \frac{1}{2}|\mu|^2 < 0$$

asymptotic stability (pathwise)

$$\lim_{t \to +\infty} |X(t)| = 0 \iff Re\left(\lambda - \frac{1}{2}\mu^2\right) < 0$$

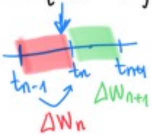
Mean-square stability of EM:

$$X_{n+1} = X_n + h\lambda X_n + \mu X_n \Delta W_{n+1}$$
$$= (1+h\lambda) X_n + \mu X_n \Delta W_{n+1}$$

$\Downarrow$

$$\mathbb{E} X_{n+1}^2 = \mathbb{E}\left[(1+h\lambda)^2 X_n^2 + 2(1+h\lambda)\mu X_n^2 \Delta W_{n+1} + \mu^2 X_n^2 \Delta W_{n+1}^2\right]$$

$$= (1+h\lambda)^2 \mathbb{E} X_n^2 + 2(1+h\lambda)\mu\, \mathbb{E}\left[X_n^2 \Delta W_{n+1}\right] + \mu^2 \mathbb{E}\left[X_n^2 \Delta W_{n+1}^2\right]$$

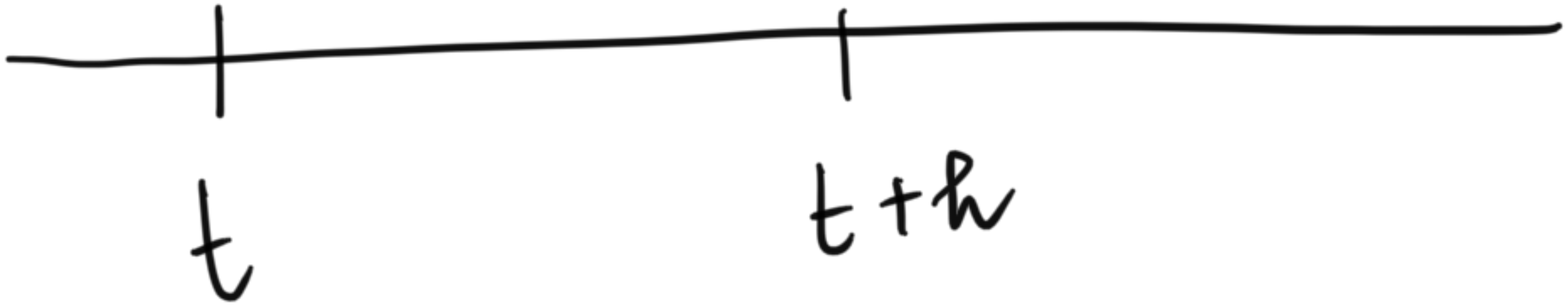


$$= (1+h\lambda)^2 \mathbb{E} X_n^2 + 2(1+h\lambda)\mu\, \mathbb{E}[X_n^2]\, \underbrace{\mathbb{E}[\Delta W_{n+1}]}_{=0} + \mu^2 \mathbb{E}[X_n^2]\, \underbrace{\mathbb{E}[\Delta W_{n+1}^2]}_{=h}$$

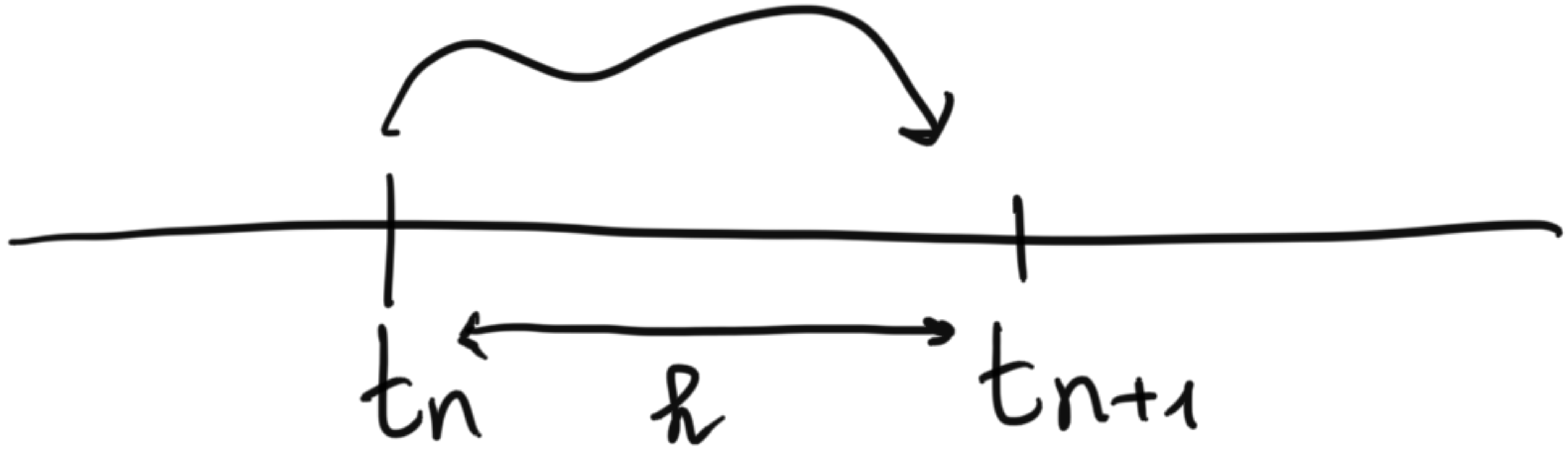
$$= \left[(1+h\lambda)^2 + h\mu^2\right] \mathbb{E} X_n^2$$

$$\lim_{t_n \to +\infty} \mathbb{E} X_{n+1}^2 = 0 \iff \boxed{(1+h\lambda)^2 + h\mu^2 < 1}$$

MEAN-SQUARE STABILITY CONDITION OF EM METHOD



$y(t+h)$



$$X_{n+1} = X_n + h\, f(t_n, X_n) + g(t_n, X_n)\left(W(t_{n+1}) - W(t_n)\right)$$

!!

EULER – MARUYAMA METHOD

Properties:

ΔW_{n+1}

1) it arises from the Itō - expansion, truncated at the first constant term;

2) one-step scheme;

3) explicit method;

Basic accuracy and stability demandings

CONSISTENCY $\longrightarrow$ local accuracy

probuty

ZERO-STABILITY → global accuracy properties
CONVERGENCE → global accuracy properties

Lax equivalence theorem (Lax-Rychtmyer theorem)   RYCHTMYER-MORTON book

convergence $\Longleftrightarrow$ consistency and zero-stability

Algorithm

$T=1$; $N=2^8$ points for the computation of Wiener increments

WIENER

```
dt = T/N;   dW = sqrt(dt) * randn(1,N);
W = cumsum(dW);
```

SDE

A standard way to proceed is the selection of a multiple of dt

$$h = R\,dt, \qquad R \geq 1$$

as stepsize of the chosen method to solve the SDE

Let us consider the SDE

$$\left\{\, dX(t) = \lambda X(t)\,dt + \mu X(t)\,dW(t), \quad \lambda, \mu \in \mathbb{R}\right.$$

$$X(0) = X_0,$$

$t \in [0, T]$.

1. Test problem for linear stability
2. asset price model; this equation is extended in the PDEs setting by the Black-Scholes.

Exact solution: $$X(t) = X_0 \exp\left(\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu W(t)\right).$$

$$W(t_j) - W(t_{j-1}) = W(jh) - W((j-1)h)$$

$$= W(jR\,dt) - W((j-1)R\,dt)$$

$$= \sum_{k=1}^{jR} dW_k - \sum_{k=1}^{(j-1)R} dW_k$$

$$= \sum_{k=(j-1)R+1}^{jR} dW_k$$

A first order truncation of the Itō-Taylor expansion we considered last time leads to the following scheme:

MILSTEIN SCHEME

$$X_{n+1} = X_n + h\, f(t_n, X_n) + g(t_n, X_n)\, \Delta W_{n+1}$$
$$+ \frac{1}{2} g(t_n, X_n)\, g'(t_n, X_n) \left(\Delta W_{n+1}^2 - \Delta t\right)$$

supposing that your SDE is scalar.

Spoiler : "order" STRONG

EM	1/2
MIL	1

Strong and weak convergence

For complete proofs of classical

results on the convergence of stochastic one-step methods,
→ GARD monograph

pathwise

in mean (mean-square)

two possibilities:

1) check that the average error tends to 0 as $h \to 0$;

strong order of convergence

$$\sup_{n=1,\dots,L} \mathbb{E}\,|X_n - X(t_n)| = O(h^{\gamma})$$

strong convergence

2) check that the average of the numerical solution is close enough to the exact average

$$\sup_{n=1,\dots,L} |\mathbb{E}(X_n) - \mathbb{E}X(t_n)| = O(h^{\tau})$$

WEAK ORDER OF CONVERGENCE

$$\sup_{n=1,\dots,L} \left|\mathbb{E}(p(X_n)) - \mathbb{E}\left(p(X(t_n))\right)\right| = O(h^{\tau}),$$

WEAK CONVERGENCE

when p is a smooth enough function

(typically p satisfies a polynomial growth ([illegible]), φ satisfies polynomial [illegible] condition)

	STRONG	WEAK
EM	1/2	1
MILSTEIN	1	1
	3/2	2

Rössler