Linear stability issues

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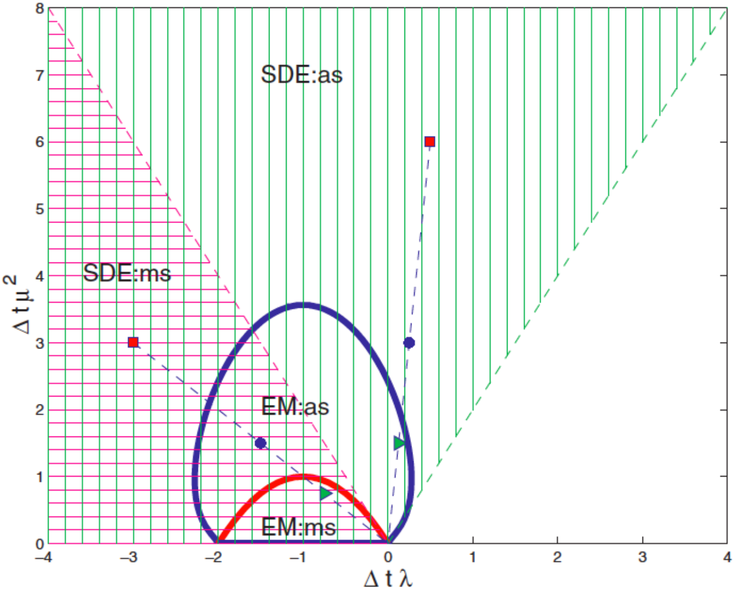
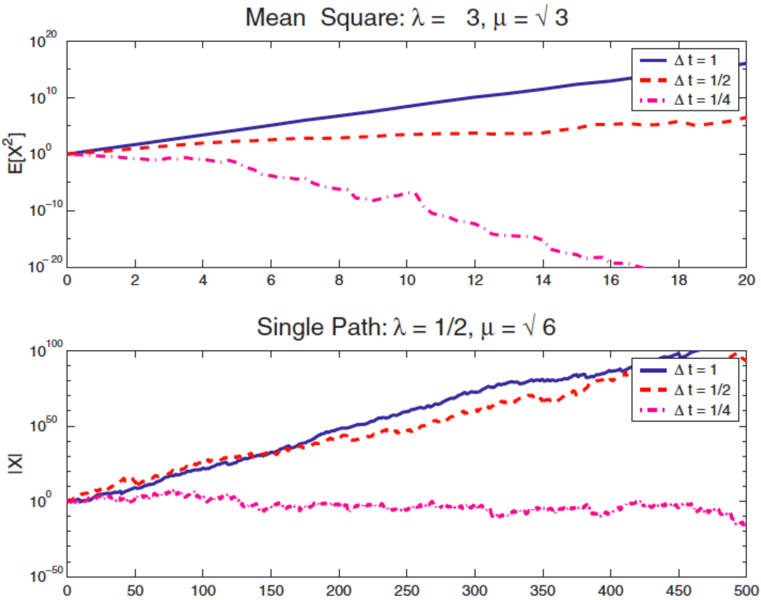
Stability regions: Euler-Maruyama



Numerical evidence



Improve stability regions

Stochastic ϑ-methods:

$$X_{n+1} = X_n + (1-\vartheta)\Delta t f(t_n, X_n) + \vartheta \Delta t f(t_{n+1}, X_{n+1}) + \sqrt{\Delta t} g(t_n, X_n) V_n,$$

with $\vartheta \in [0,1]$ and V_n standard normal random variable.

E. Buckwar, T. Sickenberger, *A comparative linear mean-square stability analysis of Maruyama- and Milstein-type methods*, Math. Comput. Simul., 81 (2011), pp. 1110–1127.

D. Higham, *Mean-square and asymptotic stability of the stochastic theta method*, SIAM J. Numer. Anal., 38 (2000), pp. 753–769.

P. E. Kloeden and E. Platen, *Numerical solution of stochastic differential equations*, Springer-Verlag, Berlin, 1992.

Y. Saito and T. Mitsui, *Stabilty analysis of numerical schemes for stochastic differential equations*, SIAM J. Numer. Anal., 33 (1996), pp. 333–344.

- One-parameter family of one-step methods;
- the value of ϑ can be chosen to enlarge the stabilty region of EM;
- implicit methods for $\vartheta \neq 0$.

Improve stability regions (ctd.)

Applying stochastic ϑ-methods to the stochastic Dahlquist test equation

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t),$$

with $\lambda, \mu \in \mathbb{C}$, yields

$$X_{n+1} = (a + bV_n)X_n,$$

with

$$a := \frac{1 + (1-\vartheta)\Delta t\lambda}{1 - \vartheta\Delta t\lambda}, \quad b := \frac{\sqrt{\Delta t}\mu}{1 - \vartheta\Delta t\lambda}.$$

Then,

$$\mathbb{E}|X_{n+1}|^2 = (|a|^2 + |b|^2)\mathbb{E}|X_n|^2.$$

As a consequence,

$$\lim_{n\to\infty} \mathbb{E}|X_n|^2 = 0 \Leftrightarrow |a|^2 + |b|^2 < 1.$$

Therefore, a stochastic ϑ-method is mean-square stable if and only if

$$\frac{|1 + (1-\vartheta)\Delta t\lambda| + \Delta t|\mu|^2}{|1 - \vartheta\Delta t\lambda|} < 1.$$

Improve stability regions (ctd.)

In summary,

$$S_{SDE} = \left\{ \lambda, \mu \in \mathbb{C} : \mathrm{Re}(\lambda) + \frac{1}{2}|\mu|^2 < 0 \right\},$$

$$S_{STM} = \left\{ x, y \in \mathbb{C} : \frac{|1 + (1-\vartheta)x| + |y|^2}{|1 - \vartheta x|} < 1 \right\},$$

with $x = \lambda \Delta t$ and $y = \mu\sqrt{\Delta t}$. From Higham (2000):

THEOREM 4.1. *For all $\Delta t > 0$ we have*

$$\begin{aligned} &S_{\mathrm{STM}}(\theta, \Delta t) \subset S_{\mathrm{SDE}} \quad \text{for } 0 \le \theta < \tfrac{1}{2}, \\ &S_{\mathrm{STM}}(\tfrac{1}{2}, \Delta t) \equiv S_{\mathrm{SDE}}, \\ &S_{\mathrm{STM}}(\theta, \Delta t) \supset S_{\mathrm{SDE}} \quad \text{for } \tfrac{1}{2} < \theta. \end{aligned}$$

For $0 \le \theta < \frac{1}{2}$, given $(\lambda, \mu) \in S_{\mathrm{SDE}}$, the STM is mean-square stable if and only if

$$\Delta t < \frac{-2(\Re\{\lambda\} + \frac{1}{2}|\mu|^2)}{|\lambda|^2(1-2\theta)}. \tag{4.3}$$

Improve stability regions (ctd.)

In other terms,

- if $0 \leq \vartheta < \frac{1}{2}$, the method is stable under stepsize restrictions (they may be severe);
- if $\vartheta = \frac{1}{2}$ (stochastic trapezoidal method), the stability region of the method coincides with that of the problem;
- if $\vartheta > \frac{1}{2}$, the stability region of the method contains that of the problem and the method is stable for any choice of the stepsize.

Then, for $\frac{1}{2} \leq \vartheta \leq 1$, we have the mean-square generalization of the deterministic A-stability.