# TA on spectrum, resolvent identities

trace-class operators,

isometry and unitary operators

# RESOLVENT and SPECTRUM

**Def** $z \in \mathbb{C}$, is an eigenvalue of $A, D(A)$

if $\exists\ \psi \in D(A)$, $\psi \neq 0$ s.t. $A\psi = z\psi$

$\Longleftrightarrow$ $z \in \mathbb{C}$, eigenv. of $A, D(A)$

$\ker(A-z) \neq \{0\}$ $\quad$ $\psi \in \ker(A-z)$, $\psi \neq 0$

is the eigenv. of $A$ corr. to $z$

If $z$ is not an eigenv., $\ker(A-z) = \{0\} \Rightarrow A-z$ is invertible. It may happen that $(A-z)^{-1} = R_A(z)$ is not bounded $\longrightarrow$ is not defined in the whole $\mathcal{H}$

L2

$\forall \{f_n\}, f_n \in D(A)\ f_n \to f,\ Af_n \longrightarrow g$
one has $f \in D(A)$ and $Af = g$

**Def.** let $A, D(A)$ be a closed operator. then

$\rho(A) = \{ z \in \mathbb{C} : (A-z)^{-1}$ is a bounded op in $\mathcal{H} \}$

resolvent set of $A, D(A)$

the spectrum of $A$ is $\sigma(A) = \mathbb{C} \setminus \rho(A)$

$z$ e.g. $\Rightarrow z \in \sigma(A)$

$\not\Leftarrow$ in the $\infty$-dim. case the spectrum contains more than eigenvalues

**Prop** (resolvent identity)

Let $A, D(A)$ be a closed op. and $z, w \in \rho(A)$. Then

$$(A-z)^{-1} - (A-w)^{-1} = (z-w)\,(A-z)^{-1}(A-w)^{-1} \qquad (*)$$

Moreover $\rho(A)$ is open and therefore $\sigma(A)$ is closed.

**Proof** of $(*)$

$$(A-z)^{-1} - (z-w)\,(A-z)^{-1}(A-w)^{-1}$$

$$= (A-z)^{-1}\left[\mathbb{1} - \overset{\pm A}{(z-w)}\,(A-w)^{-1}\right]$$

$$= (A-z)^{-1}\left[\mathbb{1} + \left[(A-z) - (A-w)\right](A-w)^{-1}\right]$$

$$= \cancel{(A-z)^{-1}} + (A-w)^{-1} - \cancel{(A-z)^{-1}} \qquad \blacksquare$$

Show $\rho(A)$ is open. Fix $z_0 \in \rho(A)$ and $D_0 = \{ z \in \mathbb{C} : |z - z_0| \, \|(A - z_0)^{-1}\| < 1 \}$

If $z \in D_0$, the sequence of bounded op.

$$R_n = \sum_{k=0}^{n} (z - z_0)^k \left[ (A - z_0)^{-1} \right]^{k+1} \qquad A^{-1} : \mathcal{H} \to D(A)$$

converges in norm to a bounded op. $R$. Show: $R = (A - z)^{-1}$.

$\forall f \in \mathcal{H}$, $R_n f \in D(A)$ and $R_n f \xrightarrow[n \to +\infty]{} Rf$

$$A R_n f = (A - z_0) R_n f + z_0 R_n f$$

$$= \sum_{k=0}^{n} (z - z_0)^k \left[ (A - z_0)^{-1} \right]^k f + z_0 R_n f$$

$k=0$ $\quad$ $j = k-1$

$$= f + \underbrace{\sum_{j=0}^{n-1} (z - z_0)^{j+1} \left[ (A - z_0)^{-1} \right]^{j+1} f}_{(z - z_0) R_n f} + z_0 R_n f \xrightarrow[n \to +\infty]{} f + zRf$$

Since $A$ is closed $(A - z) Rf = f$.

let $f \in D(A)$

$$R_n A f = R_n (A - z_0) f + z_0 R_n f$$

$$= \sum_{k=0}^{n} (z - z_0)^k \left[ (A - z_0)^{-1} \right]^k f + z_0 R_n f$$

$$= f + (z - z_0) R_{n-1} f + z_0 R_n f \longrightarrow f + z R f$$

$\Rightarrow$ $R(A-z)f = f$ Together with $(A-z)Rf = f$ we conclude that $R = (A-z)^{-1}$.

Since $\forall z \in D_0$ $R = (A-z)^{-1}$ bouded hence $z \in \rho(A)$

$\Rightarrow$ $\rho(A)$ is open $\Rightarrow$ $\sigma(A)$ is closed

$\boxtimes$

**Prop.** (second resolvent identity) Let $A, D(A)$ be s.a. and $B, D(B)$ symmetric and small perturb. w.r.t. $A$ ($\|B\phi\| \le a\|A\phi\| + b\|\phi\|$, $a \in (0,1)$, $b > 0$). For $z \in \rho(A) \cap \rho(A+B)$

$$(A+B-z)^{-1} = (A-z)^{-1} - (A-z)^{-1} B (A+B-z)^{-1}$$
$$= (A-z)^{-1} - (A+B-z)^{-1} B (A-z)^{-1} \leftarrow$$

**Proof** The hp. on $A, D(A)$, $B, D(B)$ ensure that $\forall z \in \rho(A) \Rightarrow B(A-z)^{-1}$ is bounded, $\forall z \in \rho(A+B) \Rightarrow B(A+B-z)^{-1}$ is bounded (check!). Then

$$B(A+B-z)^{-1} = (A+B-z)(A+B-z)^{-1} - (A-z)(A+B-z)^{-1}$$
$$= \mathbb{1} - (A-z)(A+B-z)^{-1}$$

$$\Rightarrow (A-z)^{-1} B (A+B-z)^{-1} = (A-z)^{-1} - (A+B-z)^{-1}$$

∎

<u>Prop</u> let $A, D(A)$ a symmetric operator. then the corresp. eigenvalues* are real and the eigenv. corresp. to different eigenvalues are orthogonal.

* no info on the other parts of the spectrum

<u>Proof</u> If $f \in \ker(A-z)$, $\operatorname{Im} z \neq 0 \Rightarrow f=0$

$$z\langle f,f\rangle \overset{Af=zf}{=} \langle f, Af\rangle \overset{\text{sym}}{=} \langle Af, f\rangle = \bar{z}\langle f,f\rangle$$

If $z \neq \bar{z} \Longrightarrow f=0 \Longrightarrow$ eigenvalues are real.

Assume now $Af=\lambda f$, $Ag=\mu g$ $\lambda\neq\mu$. Then

$$\lambda\langle g,f\rangle = \langle g, Af\rangle \underset{\text{sym.}}{=} \langle Ag, f\rangle = \mu\langle g,f\rangle$$

$$\mu\neq\lambda \Longrightarrow \langle g,f\rangle = 0$$

**Prop.** Let $A, D(A)$ be self-adjoint $\Rightarrow \sigma(A) \subseteq \mathbb{R}$

**Proof** If $A, D(A)$ is self-adjoint for any $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$ the operator $(A-z)^{-1} : \mathcal{H} \to D(A)$ is bounded and satisfies

$$\| (A-z)^{-1} \| \leq \frac{1}{|\operatorname{Im} z|} \qquad (*)$$

$\Rightarrow z : \operatorname{Im} z \neq 0$ belongs to $\rho(A) \Rightarrow \sigma(A) \subseteq \mathbb{R}$.

**Proof of $(*)$** Recall $A, D(A)$ s.a. $\forall z \in \mathbb{C}, \operatorname{Im} z \neq 0$

$\underline{\operatorname{Ran}(A-z) = \mathcal{H}}$ and $\underline{\ker(A-z) = \{0\}}$ $\hookrightarrow (A-z)^{-1}$ well defined

$\forall f \in \mathcal{H}, \exists h \in D(A)$ s.t. $f = (A-z)h$

$$\| (A-z) h \|^2 = \langle (A - \operatorname{Re} z - i \operatorname{Im} z) h, (A - \operatorname{Re} z - i \operatorname{Im} z) h \rangle$$

$$= \| (A - \operatorname{Re} z) h \|^2 + (\operatorname{Im} z)^2 \| h \|^2 \geq (\operatorname{Im} z)^2 \| h \|^2 \quad \checkmark$$

**COMPACT OPER.** A bounded op. in $\mathcal{H}$ is compact if for every bounded sequence $\{f_n\}$ in $\mathcal{H}$, $\{Af_n\}$ has a subsequence convergent in $\mathcal{H}$

- every compact op. is the norm limit of a sequence of operators of finite rank
- $\{T_n\}$ compact and $T_n \to T$ in norm $\Rightarrow$ $T$ is compact
- a bounded op. is compact $\Leftrightarrow$ its adjoint is compact.

Prop. Let $A$ be compact op. on $\mathcal{H}$, then $\sigma(A)$ is a discrete set having no limit points except eventually $\lambda = 0$. Moreover any non zero element of $\sigma(A)$ is an eigenvalue of finite multiplicity

**Prop.** Let $A$ be compact op. on $\mathcal{H}$. Then exist an orthonormal sets $\{\psi_n\}_{n=1}^{N}$ and $\{\phi_n\}_{n=1}^{N}$ (non necessarily complet) and some positive real numbers $\{\lambda_n\}_{n=1}^{N}$ (singular values of the operator) with $\lambda_n \to 0$ so that

the proof shows that $\lambda_n$ are the eigenvalues of $|A| = \sqrt{A^*A}$

$$A = \sum_{n=1}^{N} \lambda_n \langle \psi_n, \cdot \rangle \phi_n$$

where the sum (which may be finite or infinite) converges in norm.

**IDEA of the proof** Since $A$ is compact, $A^*A$ is compact and s.a. Use the eigenfunction expansion theorem and the fact that $A^*A\psi_n = \mu_n \psi_n$ with $\mu_n > 0$ [Reed-Simon vol I, thm VI.17]

**TRACE of an operator** let $\{\varphi_n\}$ is an ONB in $\mathcal{H}$. then for any positive operator $A$ we define

$$\operatorname{Tr} A = \sum_{n=1}^{\infty} \langle \varphi_n, A\varphi_n \rangle \qquad \text{Trace of } A$$

<u>Prop.</u> i) $\operatorname{Tr} A$ is independent on the basis chosen

ii) $\operatorname{Tr}(A+B) = \operatorname{Tr} A + \operatorname{Tr} B$

iii) $\operatorname{Tr}(\lambda A) = \lambda \operatorname{Tr} A \quad \forall \lambda \geq 0$

iv) $\operatorname{Tr}(U A U^{-1}) = \operatorname{Tr} A \quad \forall U$ unitary

v) If $0 \leq A \leq B$ then $\operatorname{Tr} A \leq \operatorname{Tr} B$

**Def.** An operator $A$ is trace class

$\iff \operatorname{Tr}|A| < \infty \qquad |A| = \sqrt{A^* A}$

Trace class operators describe "mixed states in QM"

**Prop.** • the family of all trace class operators is a vector space (denoted with $\mathcal{L}_1$)

• trace class operators are ideal with w.r. to bounded op. ie. $A$ trace-class, $B$ bounded

then $AB$ and $BA$ are trace-class

• If $A$ is trace class $\Rightarrow$ $A^*$ is trace class

# TRACE CLASS vs COMPACT oper.

**Prop.** Every trace class op. is compact.

A compact op. is trace class $\Longleftrightarrow$ $\sum_{n=1}^{\infty} \lambda_n < \infty$

with $\{\lambda_n\}_{n=1}^{\infty}$ singular values of A

**Proof.** If $\mathrm{Tr}|A| < \infty \Rightarrow \mathrm{Tr}|A|^2 < \infty$, $\{\varphi_n\}$ BON in $\mathcal{H}$

$$\mathrm{Tr}(|A|^2) = \sum_{n=1}^{\infty} \langle \varphi_n, A^* A \varphi_n \rangle = \sum_{n=1}^{\infty} \| A\varphi_n \|^2 < \infty \quad \forall \{\varphi_n\}$$

Suppose $\psi \in \{\varphi_1, \dots, \varphi_N\}^{\perp}$ and $\|\psi\| = 1$

$$\|A\psi\|^2 \leq \mathrm{Tr}|A|^2 - \sum_{n=1}^{N} \|A\varphi_n\|^2$$

$\sum_{n=1}^{N} \langle \varphi_n, \cdot \rangle A\varphi_n$ converges in norm to A

$\Rightarrow$ A is compact

$$\Rightarrow \sup \{ \|A\psi\| \text{ s.t. } \psi \in \{\varphi_1, \dots, \varphi_N\}^{\perp}, \|\psi\| = 1 \} \xrightarrow[N \to +\infty]{} 0$$

b) $A$ compact, $\sum_{n=1}^{\infty} \lambda_n < \infty \Rightarrow A$ trace class

is proved using the canonical form for compact op

Rk$_1$ Canonical form for trace-class op. $A = \sum_{n=1}^{\infty} \lambda_n \langle \psi_n, \cdot \rangle \psi_n$

Rk$_2$ the finite rank operators are $\underbrace{\|\cdot\|_1}_{\text{trace-class norm}}$-dense in $\mathcal{L}_1$

## HILBERT-SCHMIDT op. (HS)

An op. $T$ is called HS if $\operatorname{Tr} T^* T < \infty$

Note that: $\|A\| \leq \underbrace{\|A\|_2}_{\text{HS norm}} \leq \|A\|_1$

## CARACTERIZATION of the essential spectrum $\sigma_{ess} = \sigma \setminus \sigma_d$



**Prop.** $\lambda \in \sigma_{ess}(A)$ iff. $\exists$ a singular Weyl sequence relative to $\lambda$ ie a sequence of vectors $\{f_n\}$ s.t.

$f_n \in D(A)$, $\|f_n\| = 1$, $f_n \rightharpoonup 0$ for $n \to +\infty$

and

$$\|(A-\lambda) f_n\| \xrightarrow[n \to +\infty]{} 0$$

**Prop.** $\lambda \in \sigma(A)$ iff. $\exists$ a Weyl sequence relative to $\lambda$

**Proof**
- $\exists \{f_n\}$ Weyl seq. rel. to $\lambda \Rightarrow \lambda \in \sigma(A)$ (proof by absurd)
- $\lambda \in \sigma(A) \Rightarrow \exists$ Weyl seq. One builds the Weyl sequence

by using $z_n = \lambda + \frac{i}{n}$, $z_n \in \rho(A)$: $|z_n - \lambda| \to 0$, $\|(A-z_n)^{-1}\| \to +\infty$

$\Rightarrow \exists \{g_n\}$: $\|g_n\| = 1$ and $\|(A-z_n)^{-1} g_n\| \to \infty \Rightarrow$ $f_n = \dfrac{(A-z_n)^{-1} g}{\|(A-z_n)^{-1} g\|}$ is Weyl seq. for $A$ rel. to $\lambda$

Weyl's thm let $A, D(A)$ and $B, D(B)$ s.a. op. in $\mathcal{H}$

and

$$R_A(z) = (A-z)^{-1} \qquad R_B(z) = (B-z)^{-1}$$

If $\exists$ $z \in \rho(A) \cap \rho(B)$ s.t. $R_A(z) - R_B(z)$ is a compact op $\Rightarrow$ $\sigma_{ess}(A) = \sigma_{ess}(B)$

<u>Proof of Weyl theorem</u>. $\lambda \in \sigma_{ess}(A) \Rightarrow \lambda \in \sigma_{ess}(B)$

(the viceversa is proven analogously)

let $\lambda \in \sigma_{ess}(A) \Rightarrow \{\psi_n\}$ a singular Weyl sequence for A rel. to $\lambda$

$$\psi_n \in D(A), \quad \|\psi_n\| = 1, \quad \psi_n \rightharpoonup 0 \; : \; \|(A-\lambda)\psi_n\| \longrightarrow 0$$

then

$$\left(R_A(z) - \frac{1}{\lambda - z}\right)\psi_n = \left(R_A(z) - \frac{1}{\lambda - z}\overbrace{R_A(z)(A-z)}^{\mathbb{1}}\right)\psi_n$$

$$= R_A(z)\left(\mathbb{1} - \frac{A-z}{\lambda - z}\right)\psi_n = \frac{R_A(z)}{\lambda - z}(A-\lambda)\psi_n$$

$$\Rightarrow \left\|\left(R_A(z) - \frac{1}{\lambda - z}\right)\psi_n\right\| \le \frac{\|R_A(z)\|}{|\lambda - z|}\underbrace{\|(A-\lambda)\psi_n\|}_{\downarrow\, n\to+\infty \atop 0}$$

let us define $\phi_n = R_B(z)\psi_n$ ; we have $\phi_n \in D(B)$ and

$\phi_n \rightharpoonup 0$ (since $\psi_n \rightharpoonup 0$). let us show $\|\phi_n\| \not\to 0$

Write $\phi_n = (R_B(z) - R_A(z))\psi_n + \left(R_A(z) - \frac{1}{\lambda - z}\right)\psi_n + \frac{1}{\lambda - z}\psi_n$

$$\lim_n \|\phi_n\| = \underbrace{\lim_n \|(R_B(z)-R_A(z))\psi_n\|}_{=0 \text{ by ass.}} + \underbrace{\lim_n \left\|R_A(z) - \frac{1}{\lambda - z}\right\|}_{=0 \text{ by comp. bef.}} + \frac{1}{|\lambda - z|}$$

$\|\psi_n\| = 1$

With $\hat{\phi}_n = \frac{\phi_n}{\|\phi_n\|}$ we get $\|\hat{\phi}_n\| = 1$

It remains to show that $\|(B-\lambda)\hat{\phi}_n\| \longrightarrow 0$.

$\|(B-\lambda)\phi_n\| = \|(z-\lambda)\phi_n + (B-z)\phi_n\|$ with $\phi_n = R_B(z)\psi_n$

$= |z-\lambda| \left\|\left(R_B(z) - \frac{1}{\lambda - z}\right)\psi_n\right\|$

$\leq |z-\lambda| \left( \underbrace{\|(R_B(z) - R_A(z))\psi_n\|}_{\to 0} + \underbrace{\left\|\left(R_A(z) - \frac{1}{\lambda - z}\right)\psi_n\right\|}_{\to 0} \right) \Rightarrow \lambda \in \sigma_{ess}(B)$

# UNITARY OPERATORS

Consider $\mathcal{H}, \langle\cdot,\cdot\rangle, \|\cdot\|, \mathbb{1}$

$\mathcal{H}', \langle\cdot,\cdot\rangle', \|\cdot\|', \mathbb{1}'$

Def. An operator $T: \mathcal{H} \to \mathcal{H}'$ is an ISOMETRY if it preserves the scalar product ie $\langle Tf, Tg\rangle' = \langle f, g\rangle$ $\forall f, g \in \mathcal{H}$ $\iff$ $T^* T = \mathbb{1}$

where $T^*$ def. by $\langle f', Tg\rangle' = \langle T^* f', g\rangle$ $\forall f' \in \mathcal{H}', g \in \mathcal{H}$

RK If $T$ is isometric $\|Tf\| = \|f\| \Rightarrow \|T\| = 1$, $\mathrm{Ker}\, T = \{0\}$

$\Rightarrow T^{-1}$ exists but in general is not defined in the whole $\mathcal{H}$

**Def.** A operator $U : \mathcal{H} \to \mathcal{H}'$ is unitary if it is an isometry and $\mathrm{Ran}(U) = \mathcal{H}'$

**Prop.** A bounded op. $U : \mathcal{H} \to \mathcal{H}'$ is unitary iff $U^* U = \mathbb{1}$ and $U U^* = \mathbb{1}' \iff U^* = U^{-1}$

**Example** An operator which is isometric but not unitary is

$$D(T) = \ell^2 = \left\{ \{x_n\}_{n \in \mathbb{N}},\ x_n \in \mathbb{C} : \sum_{x=1}^{\infty} |x_n|^2 < \infty \right\}$$

$$T : (x_1, \dots, x_n) \longrightarrow (0, x_1, \dots x_n) \quad \text{shift op.}$$

**Check.** $T^* : (x_1, \dots x_n) \longrightarrow (x_2, \dots, x_n)$

$$T^* T = \mathbb{1} \quad \text{but} \quad T T^* \neq \mathbb{1}$$

<u>Prop</u>. Let $A, D(A)$ is a s.a. operator in $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}'$ is unitary. Then the operator in $\mathcal{H}'$

$$D(A') = \{ f' \in \mathcal{H}' : f' = Uf , \ f \in D(A) \}$$

$$A' f' = U A U^{-1} f' : D(A') \longrightarrow \mathcal{H}'$$

is self-adjoint, $\sigma(A) = \sigma(A')$ and eigenvalues of $A$ and $A'$ coincide.

<u>Proof</u> a) $A'$ is symmetric

$$\langle g', A'f' \rangle' \overset{\text{def of } A}{=} \langle g', UAU^{-1}f' \rangle' = \langle U^{-1}g', AU^{-1}f' \rangle$$

$$\overset{\text{s.a. of } A}{=} \langle UAU^{-1}g', f' \rangle' = \langle A'g', f' \rangle' \quad \blacksquare$$

b) $A'$, $D(A')$ is self-adjoint

Fix $z \in \rho(A) \Longrightarrow (A-z)^{-1}$ is bounded on $\mathcal{H}$

$f' \in \mathcal{H}'$ and define $u' = U \underbrace{(A-z)^{-1} \overbrace{U^{-1} f'}^{\mathcal{H}}}_{D(A)} \in D(A')$

Moreover

$$(A'-z)u' \overset{\text{def of } A'}{=} U(A-z)\underbrace{U^{-1}U}_{\mathbb{1}}(A-z)^{-1}U^{-1}f' = f' \quad (*)$$

$\Longrightarrow \operatorname{ran}(A'-z) = \mathcal{H}$

since $\forall f' \in \mathcal{H}' \ \exists u'$ s.t. $(A'-z)u' = f'$

$\Longrightarrow$ self-adj. criterion implies $A'$ self-adjoint ✱

c) $\rho(A) \subseteq \rho(A')$

$z \in \rho(A) \Rightarrow (A-z)^{-1}$ bounded

$\Rightarrow U(A-z)^{-1}U^{-1} ; \mathcal{H}' \longrightarrow D(A')$ is bounded

On the other hand from (*) $(A-z)\, U(A-z)^{-1}U^{-1} f' = f' \quad \forall f' \in \mathcal{H}'$

hence $U(A-z)^{-1}U^{-1} = (A'-z)^{-1}$ bounded $\Rightarrow z \in \rho(A')$

Changing the roles of $A$ and $A'$ one also find $\rho(A') \subseteq \rho(A)$

$\Rightarrow \rho(A') = \rho(A) \quad \blacksquare$