TA on self-adjoint operators

ADJOINT OPERATOR

let A is a bounded op in $\mathcal{H}$. For any $f \in \mathcal{H}$

$$\mathcal{J}_f : \underset{D(A)}{g \in \mathcal{H}} \longrightarrow \langle f, Ag \rangle$$

the linear funct is bouded

$$|\mathcal{J}_f(g)| = |\langle f, Ag \rangle| \leq \|f\| \, \underset{\text{boude}}{\|A\|} \, \|g\| \quad \checkmark$$

By Riesz thm $\exists!\ h \in \mathcal{H}$ s.t.

$$\mathcal{J}_f(g) = \langle f, Ag \rangle = \langle h, g \rangle \qquad \forall g \in D(A) = \mathcal{H}$$

then $A^* : f \longrightarrow A^* f = h \ : \ \langle f, Ag \rangle = \langle A^* f, g \rangle$

Def let A bounded op. The adjoint of A is

$A^*: \mathcal{H} \to \mathcal{H}$ s.t. $\langle f, Ag\rangle = \langle A^* f, g\rangle \quad \forall g \in \mathcal{H}$

Exercise Show A^* is linear, bounded $\|A^*\| = \|A\|$

$$\|A^* f\| = \sup_{g\in\mathcal{H}, \|g\|=1} |\langle g, A^* f\rangle|$$

$$= \sup_{g\in\mathcal{H}, \|g\|=1} |\langle Ag, f\rangle| \leq \|f\| \overbrace{\sup_{g \ldots} \|Ag\|}^{\|A\|}$$

$$\Rightarrow \quad \|A^*\| \leq \|A\|$$

Similarly one shows $\|A\| \leq \|A^*\|$.

Example of adjoints of bounded op:

- $(M_\varphi f)(x) = \varphi(x) f(x)$ $\qquad M_\varphi^* = M_{\overline{\varphi}}$
- K s.t. $(Kf)(x) = \int k(x,y) f(y)\, dy$ $\Rightarrow (K^* f)(x) = \int \overline{k(y,x)} f(y)\, dy$
  with $k \in L^2(\mathbb{R}^2)$

If $(A, D(A))$ unbounded, $\langle f, Ag \rangle$ might not be bounded for an arbitrary $f \in \mathcal{H}$.

Define $D(A^*) = \{ f \in \mathcal{H} : \ \mathcal{F}_g : g \in D(A) \longrightarrow \langle f, Ag \rangle$ is bounded $\}$

$= \{ f \in \mathcal{H} : \sup_{\substack{g \in D(A), \\ \|g\|=1}} |\langle f, Ag \rangle| < \infty \}$

By def. $\forall f \in D(A^*)$, $\mathcal{F}_f$ is bounded, hence

$\exists! \ h \in \mathcal{H}$ s.t. $\langle f, Ag \rangle = \langle h, g \rangle \ \forall g \in D(A)$

Def. $A^* : f \in D(A^*) \rightarrow h$ s.t. $\langle f, Ag \rangle = \langle h, g \rangle$ $\forall g \in D(A)$

L4

<u>Exercise</u> $\mathcal{H} = L^2(\mathbb{R})$, $Q_0: (Q_0 f)(x) = x f(x)$, $D(Q_0) = \mathcal{S}(\mathbb{R})$

<u>Show</u> $D(Q_0^*) = \{ f \in L^2(\mathbb{R}) : \int dx\, x^2 |f(x)|^2 < \infty \} \supseteq \mathcal{S}(\mathbb{R})$

$(Q_0^* f)(x) = x f(x)$

i) $f \in L^2(\mathbb{R}) : \underline{\int dx\, x^2 |f(x)|^2} < \infty \Rightarrow f \in D(Q_0^*)$

$\forall g \in D(Q_0)$ $\quad |\langle f, Q_0 g \rangle| \leq \left| \int dx\, \overline{f(x)}\, x\, g(x) \right|$

$\leq \left(\int dx\, |f(x)|^2 x^2 \right)^{1/2} \|g\|_2$ ✓

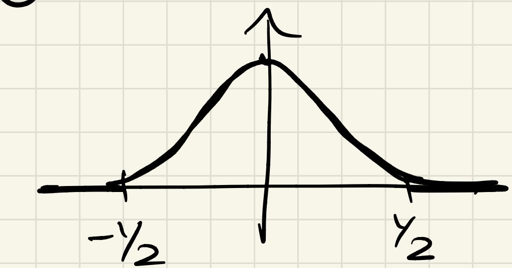
Moreover $\langle f, Q_0 g \rangle = \int dx\, (x \overline{f(x)})\, g(x) = \langle Q_0^* f, g \rangle$

ii) $f \in D(Q_0^*) \Rightarrow f \in L^2(\mathbb{R}) : \int dx\, x^2 \, |f(x)|^2 < \infty$ L5

We use an approx. argument

$$\delta(x) = \begin{cases} \dfrac{2\, e^{\frac{1}{4x^2-1}}}{\int_{-1}^{1} dy\, e^{\frac{1}{y^2-1}}} & |x| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$\int \delta(x) = 1$



then

$$\delta_N(x) = \int_{-\infty}^{x/N} \left[\delta(\underbrace{y+3/2}_{x}) - \delta(y-3/2) \right] dy \in C_0^\infty(\mathbb{R})$$

Check: $\delta_N(x)$ is non negative $\delta_N(x) = 1 \ \forall x \in [-N, N]$

$$\delta_N(x) = \int_{-\infty}^{x/N - 3/2} \delta(x)\, dx - \int_{-\infty}^{x/N + 3/2} \delta(x)\, dx = \int_{x/N - 3/2}^{x/N + 3/2} \delta(x) = \int_{-1/2}^{1/2} \delta(x) = 1$$

$\uparrow$ $x \in [0, N]$

Given $f \in D(Q_0^*)$

$\xi_N = 1 \quad x \in [-N, N]$

$$\infty > \| Q_0^* f \| = \lim_N \| \xi_N Q_0^* f \| = \lim_N \sup_{\substack{g \in D(Q_0) \\ \|g\|=1}} | \langle g, \xi_N Q_0^* f \rangle |$$

$$= \lim_N \sup_{\substack{g \in D(Q_0) \\ \|g\|=1}} | \underbrace{\langle Q_0 \xi_N g, f \rangle}_{} |$$

$$\left| \int dx \, x \, \overline{g(x)} \, \xi_N(x) \, f(x) \right|$$

$g = x \xi_N(x) f(x)$
$\in D(Q_0)$

$$\geq \lim_N \left(\int dx \, | \xi_N(x) \, x \, f(x) |^2 \right)$$

$\longrightarrow$ Monotone conv. th. implies

$$\int dx \, | x f(x) |^2 < \infty$$

∎

Exercise $\mathcal{H} = L^2(0,1)$ $(pu)(x) = -i\,u'(x)$

$$D(p) = \{ u \in H^1(0,1) : u(0) = u(1) = 0 \}$$

Verify $(p^* u)(x) = -i\,u'(x)$

$$D(p^*) = H^1(0,1)$$

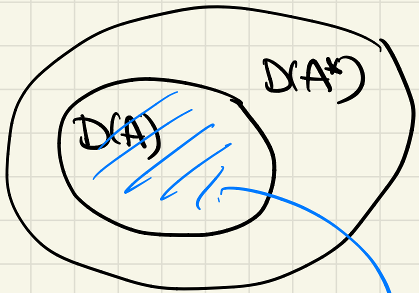
domain of the adjoint is larger

Hint p is molt. op. in $\mathcal{F}$

L8

Prop. If $(A, D(A))$ is symm, then $A^* \supseteq A$

the adjoint is an extension of A



A^* is an extension of A

$D(A^*) \supseteq D(A)$

$A^* = A \quad \forall f \in D(A)$

i) $f \in D(A) \Rightarrow f \in D(A^*)$

A symm.

$$\sup_{\substack{g \in D(A) \\ \|g\|=1}} |\langle f, Ag \rangle| \overset{\text{sym.}}{=} \sup_{g \in \cdots} |\langle Af, g \rangle| \le \|Af\| \quad \checkmark$$

ii) $f \in D(A) \Rightarrow A = A^*$

$$\langle f, Ag \rangle = \langle h, g \rangle \quad \forall g \in D(A)$$

$h = A^* f$

$$\to \langle A^* f, g \rangle \overset{\text{def of adjoint}}{=} \langle f, Ag \rangle \overset{\text{sym}}{=} \langle Af, g \rangle \leftarrow$$

Def $(A, D(A))$ is self-adjoint ($A = A^*$)

if A is symmetric and $D(A) = D(A^*)$

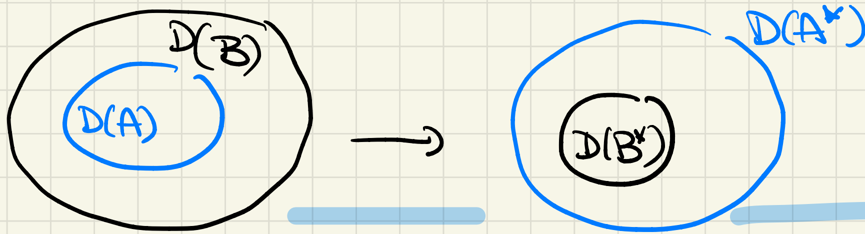
Exampl. of s.a. ope

- $M_\varphi \longrightarrow M_\varphi^* = M_{\bar{\varphi}}$ $\quad$ M_φ is self-adjoint $\iff \varphi$ is real
- K with kernel $k(x,y)$ self-adjoint $\iff k(x,y) = \overline{k(y,x)}$

Non self-adj. (symmetric operators with $D(A^*) \neq D(A)$)

- Q_0, $D(Q_0) = \mathcal{S}(\mathbb{R})$ NOT self-adjoint
  $D(Q_0^*) \supseteq D(Q_0)$
- p, $D(p)$ $L^2((0,1))$ NOT self-adjoint

Prop. let $(A, D(A))$ be a symm. op. and $(B, D(B))$ be an extension of $(A, D(A))$. then $A^*, D(A^*)$ is an extension of $B^*, D(B^*)$



Proof i) $f \in D(B^*) \Rightarrow f \in D(A^*)$

$f \in D(B^*)$

$$\sup_{g \in D(A), \|g\|=1} |\langle f, Ag \rangle| \underset{D(A) \subseteq D(B)}{\leq} \sup_{\substack{g \in D(B) \\ \|g\|=1}} |\langle f, Ag \rangle| < \infty$$

ii) $\forall \underline{f \in D(B^*)} \Rightarrow A^* = B^*$

B is an ext. of A

$$\underline{\forall g \in D(A)} \quad \langle \underline{A^* f}, g \rangle \underset{\text{def of } A^*}{=} \langle f, Ag \rangle = \langle f, Bg \rangle \underset{f \in D(B^*)}{=} \langle \underline{B^* f}, g \rangle \Longrightarrow A^* = B^*$$

L11

Def. A symm. op. is essentially self-adjoint if it admits a unique self-adjoint extension.

Prop. Let $A, D(A)$ symm. s.t. $A^*, D(A^*)$ is self adjoint then $A, D(A)$ is essentially self-adjoint and its unique self-adj. ext. is $A^*, D(A^*)$

Proof $A, D(A)$ symm. $\Rightarrow$ $A^* \supseteq A$ / $(A^*)^* = A^*$ by hp.

Let $B, D(B)$ be another self adjoint extension of $A, D(A)$

then

$$\underset{\substack{\uparrow \\ B \text{ s.a.}}}{B = B^*} \underset{\substack{\uparrow \\ \text{1st prop.}}}{\subseteq} A^*$$

On the other hand

$$B \subseteq A^* \Rightarrow \overset{B}{\overset{\parallel}{B^*}} \supseteq (A^*)^* = A^*$$

Hence $B \subseteq A^*$ & $B \supseteq A^* \Rightarrow \underline{B = A^*}$

REMARK To verify that a given symm. oper. $A, D(A)$ is self-adjoint it is sufficient to show

$$D(A^*) \subseteq D(A)$$

(we know already $D(A^*) \supseteq D(A)$)

SELF-ADJOINTNESS CRITERION

Let $A, D(A)$ be a symm. operator. If $\exists z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$ s.t.

$$\operatorname{Ran}(A-z) = \operatorname{Ran}(A-\bar{z}) = \mathcal{H}$$

then $A, D(A)$ is s.a.

Proof Goal $D(A^*) \subseteq D(A)$, $f \in D(A^*) \Rightarrow f \in D(A)$

Take $f \in D(A^*)$, $\Phi \in D(A)$

$$\langle f, (A-z)\Phi \rangle = \langle \underbrace{(A^* - \bar{z}) f}_{\text{element of } \mathcal{H}\text{, by assumption } \exists g \in D(A)}, \Phi \rangle$$

$$= \langle (A - \bar{z}) g, \Phi \rangle \underset{\text{symm.}}{=} \langle g, (A-z)\Phi \rangle$$

$$\Rightarrow \forall \Phi \in D(A) \quad \langle f - g, \underbrace{(A-z)\Phi}_{\text{covers the whole } \mathcal{H} \text{ by assumption}} \rangle = 0$$

$$\Rightarrow f - g \perp h \quad \forall h \in \mathcal{H} \qquad \Rightarrow f = g \in D(A)$$

<u>Exercise</u> Show that $(Q_0^* f)(x) = x f(x)$

$$D(Q_0^*) = \{ f \in L^2(\mathbb{R}) : \int dx \; x^2 |f(x)|^2 < \infty \}$$

(ie the adjoint of Q_0 defined on $D(Q_0) = \mathcal{S}(\mathbb{R})$) <u>is self-adj</u>

For $f \in L^2(\mathbb{R})$ consider $u_f^{\pm}(x) = \dfrac{f(x)}{x \pm i}$ (*)

i) $u_f^{\pm} \in D(Q_0^*)$ $\quad \int dx \, x^2 |u_f^{\pm}(x)|^2 = \int dx \dfrac{|f(x)|^2}{x^2+1} x^2 \leq \|f\|^2$

also any el. of $D(Q_0^*)$ can be express. as (*)

ii) $(Q_0^* \pm i) u_f^{\pm} = f$ by def. $\Rightarrow \operatorname{Ran}(Q_0^* \pm i) = L^2(\mathbb{R})$

$\Rightarrow (Q_0^*, D(Q_0^*))$ is self-adjoint

$\Rightarrow (Q_0, D(Q_0))$ admits a unique self-adj. ext. who is $(Q_0^*, D(Q_0^*))$

<u>Exercise</u> Show that the following operator in $L^2(0,1)$ is self-adjoint

$$D(P_\alpha) = \left\{ u \in H^1(0,1) : u(0) = \alpha u(1),\ \alpha \in \mathbb{C},\ |\alpha| = 1 \right\}$$

$$(P_\alpha u)(x) = -i\, u'(x)$$

<u>RK</u> For any α this is an extension $(p, D(p))$

$\to$ $(p, D(p))$ admits ∞-<u>many</u> self-adjoint extensions

<u>Prop.</u> let $A, D(A)$ is s.a. operator. then $\forall z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$ one has

$$\operatorname{Ker}(A-z) = \{0\}, \quad \operatorname{Ran}(A-z) = \mathcal{H}$$

trivial, since eigenv. of a symm. op. are reals

not only sufficent condition but also <u>necessary</u>

Moreover, $(A-z)^{-1} : \mathcal{H} \to D(A)$ is bounded and satisfies

$$\|(A-z)^{-1}\| \leq \frac{1}{|\operatorname{Im} z|}$$

<u>Proof</u> Teta's book.

KATO-RELLICH THEOREM

Let $A, D(A)$ s.a., $B, D(B)$ symm. with $D(A) \subseteq D(B)$

Assume that B is a small perturbation w.r.t. A, that is $\exists$ $a \in (0,1)$ and $b>0$ s.t.

b can be large, as long as finite

$$\|B\phi\| \leq a\|A\phi\| + b\|\phi\| \quad \forall \phi \in D(A)$$

Rk. $a=0$ corresponds to B bounded (bounded op. are a small perturb. with respect to any s.a. operators)

then the operator

$$A+B, \; D(A) \text{ is self-adjoint}$$

APPLICATION: Hydrogen atom (lecture 6)

Proof: Teta's book

Refences

Sections 4.3 $\longrightarrow$ 4.5 Teta's book