Non Interacting Particles

$$H_N = \sum_{j=1}^{N} h_j$$

$$\hookrightarrow \; h_j = \mathbb{1} \otimes \dots \otimes h \otimes \dots \otimes \mathbb{1}$$

$$e^{-itH_N} = e^{-it\sum_{j=1}^{N} h_j} = \bigotimes_{j=1}^{N} e^{-ith_j}$$

If $\varphi_{N,0} = \varphi_1 \otimes \dots \otimes \varphi_N$

$$e^{-itH} \varphi_{N,0} = e^{-ith} \varphi_1 \otimes \dots \otimes e^{-ith} \varphi_N$$

What can we say about the spectrum?

Assume h has eigenvalues $e_1, e_2, \ldots$

eigenvectors $\varphi_1, \varphi_2, \ldots$

$\varphi_{i_1} \otimes \varphi_{i_2} \otimes \ldots \otimes \varphi_{i_N}$ is an eigenvector of H_N

with eigenvalue $e_{i_1} + \ldots + e_{i_N}$

If we have bosons/fermions we can consider

$P_s^N(\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_N})$, $P_a^N(\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_N})$

Ground state?

- bosons : $\varphi_1 \otimes \dots \otimes \varphi_1 = \varphi_1^{\otimes N}$

  ground state energy : $N e_1$

- fermions $\Psi_N = \det \begin{vmatrix} \varphi_1(x_1) & \dots & \varphi_N(x_1) \\ \vdots & & \vdots \\ \varphi_1(x_N) & & \varphi_N(x_N) \end{vmatrix}$

  ground state energy: $e_1 + \dots + e_N$

$$H_N = \sum_{j=1}^{N} (-\Delta_{x_j}) \quad \text{on} \quad \Lambda = \left[-\frac{L}{2}, \frac{L}{2}\right]^3$$

eigenvalues of $-\Delta$ are p^2, $p \in \frac{2\pi}{L} \mathbb{Z}^3 = \Lambda^*$

eigenvectors are $\frac{e^{-ipx}}{\sqrt{L^3}}$

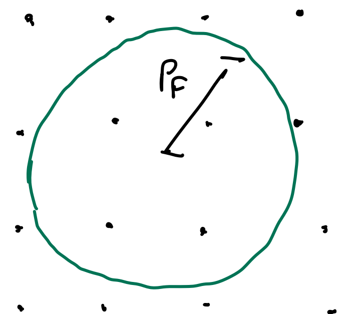
$\leadsto$ the ground state energy of the bosonic system is 0

for the fermionic system?

$$\sum_{\substack{p \in \Lambda^*: \\ p^2 < \varepsilon_F}} p^2$$

where ε_F = energy below which there are are exactly N particles

$$\rightsquigarrow \quad N = \sum_{\substack{p \in \Lambda^*: \\ p^2 < \varepsilon_F}} 1$$



p_F = Fermi momentum

$\sqrt{\varepsilon_F} = p_F$

ε_F = Fermi energy

What happens at positive Temperature ?

We work in The grand canonical ensemble

$$\rho_{GC} = \frac{e^{-\beta(\mathcal{H} - \mu \mathcal{N})}}{\mathcal{Z}_{GC}(\beta)} , \qquad \mathcal{Z}_{GC} = \mathrm{Tr}\left(e^{-\beta(\mathcal{H} - \mu \mathcal{N})}\right)$$

where $\mathcal{H} = d\Gamma(H)$ second quantization of The Hamiltonian

$\mathcal{N}$ particle number

$$\varphi_p = \frac{e^{ipx}}{\sqrt{L^3}}, \quad a_p = a(\varphi_p), \quad a_p^* = a^*(\varphi_p)$$

$$\mathcal{H} = \sum_{p \in \Lambda^*} \langle \varphi_p, H \varphi_p \rangle a_p^* a_p = \sum_{p \in \Lambda^*} p^2 a_p^* a_p$$

$$\mathcal{N} = \sum_{p \in \Lambda^*} a_p^* a_p$$

$$e^{-\beta(\mathcal{H} - \mu \mathcal{N})} = e^{\sum_p (-\beta(p^2 - \mu)) a_p^* a_p}$$

eigenvalues? $\prod_{p \in \Lambda^*} e^{\varepsilon_p n_p}$

$\varepsilon_p = -\beta(p^2 - \mu)$

$n_p = 0, 1$ for fermions

$n_p = 0, 1, 2 \ldots$ for bosons

$$e^{\sum_p \left(-\beta(p^2-\mu)\right) a_p^* a_p} = \prod_{p\in\Lambda^*} e^{-\beta(p^2-\mu) a_p^* a_p}$$

bosons

$$\hookrightarrow [a_p^* a_p, a_q^* a_q] = 0 \quad p \neq q$$

Let $p \in \Lambda^*$

$$\underline{\Psi} = (a_p^*)^{n_p} \Omega \rightsquigarrow e^{\varepsilon_p a_p^* a_p} \underline{\Psi} = e^{\varepsilon_p n_p} \underline{\Psi}$$

$$\varepsilon_p a_p^* a_p \underline{\Psi} = \varepsilon_p a_p^* \underbrace{a_p (a_p^*)^{n_p}}_{\left(n_p (a_p^*)^{n_p - 1} + \cancel{(a_p^*)^{n_p} a_p} \right)} \Omega$$

$$= \varepsilon_p a_p^* n_p (a_p^*)^{n_p-1} \Omega = n_p \varepsilon_p \underbrace{(a_p^*)^{n_p} \Omega}_{\underline{\Psi}}$$

Fermions

Analogously if we consider

$$\Psi = a_q^* \, \Omega \qquad [n_p = 1]$$

here we can move a_p on Ω and $a_p \Omega = 0$

$$\sum_{p \in \Lambda^*} \varepsilon_p a_p^* a_p \Psi = \Big(\cancel{\sum_{p \neq q} \varepsilon_p a_p^* a_p} + \varepsilon_q a_q^* a_q \Big) \Psi$$

$$= \varepsilon_q a_q^* \underbrace{a_q a_q^*}_{\left(\underbrace{\{a_q, a_q^*\}}_{1} - \cancel{a_q^* a_q} \right) \Omega} \Omega = \varepsilon_q a_q^* \Omega = \varepsilon_q n_q \Psi$$

⇝ also in this case the spectrum is given by $\prod_p e^{\varepsilon_p n_p}$ where $n_p \in \{0, 1\}$

$$Z_{GC} = \mathrm{Tr}\left(e^{-\beta(\mathcal{H}-\mu\mathcal{N})}\right) = \sum_{\substack{n_p \\ p\in\Lambda^*}} \prod_{p\in\Lambda^*} e^{\varepsilon_p n_p}$$

$n_p = 0, 1$ f

$n_p = 0, 1, 2, \dots$ b

For bosons

$$\prod_{p\in\Lambda^*} \underbrace{\sum_{n_p} e^{\varepsilon_p n_p}}_{\text{geometric sum}} = \prod_{p\in\Lambda^*} \frac{1}{1-e^{\varepsilon_p}}, \qquad \varepsilon_p = -\beta(p^2-\mu)$$

⚠ To have convergence we have to ask $\varepsilon_p < 0 \quad \forall p \in \Lambda^*$

$\mu < 0$

For fermions

$$\prod_{p\in\Lambda^*} \sum_{n_p} e^{\varepsilon_p n_p} = \prod_{p\in\Lambda^*} \left(1 + e^{\varepsilon_p}\right)$$

$\hookrightarrow n_p \in \{0, 1\}$

we have just the sum of two terms

$$\Omega(\beta,\mu) = -\frac{1}{\beta}\log Z_{GC}(\beta,\mu) = \begin{cases} \dfrac{1}{\beta}\displaystyle\sum_{p\in\Lambda^*}\log\left(1-e^{\varepsilon_p}\right) & B \\[2ex] -\dfrac{1}{\beta}\displaystyle\sum_{p\in\Lambda^*}\log\left(1+e^{\varepsilon_p}\right) & F \end{cases}$$

$$N = \overset{\mathrm{Tr}(\mathcal{N}\rho_{GC})}{\overset{\shortparallel}{\langle \mathcal{N}\rangle}} = -\frac{\partial\Omega(\beta,\mu)}{\partial\mu} = \sum_{p\in\Lambda^*} \underset{\sim\,\langle\mathcal{N}_p\rangle}{n(\varepsilon_p)}$$

where

$$n(\varepsilon_p) = \begin{cases} \dfrac{1}{e^{-\varepsilon_p}-1} & \text{bosons} \\[2ex] \dfrac{1}{e^{-\varepsilon_p}+1} & \text{fermions} \end{cases}$$

<u>claim</u> $n(\varepsilon_p) = \langle \mathcal{N}_p \rangle$ where $\mathcal{N}_p = a_p^* a_p$

$$\left[\mathcal{N} = \sum_p \mathcal{N}_p \right]$$

Indeed

$$\langle \mathcal{N}_p \rangle = \mathrm{Tr}\left(\rho_{GC} \mathcal{N}_p \right) = \frac{\sum_{n_p} n_p e^{\varepsilon_p n_p} \prod_{\substack{q \in \Lambda^* \\ q \neq p}} \sum_{n_q} e^{\varepsilon_q n_q}}{\prod_{q \in \Lambda^*} \sum_{n_q} e^{\varepsilon_q n_q}}$$

$$= \frac{\sum_{n_p} n_p e^{\varepsilon_p n_p}}{\sum_{n_p} e^{\varepsilon_p n_p}} = -\frac{\partial}{\partial x} \log\left(\sum_n e^{-nx} \right) \Bigg|_{x = \beta(p^2 - \mu)}$$

$$= n(\varepsilon_p)$$

$$E = \langle \mathcal{H} \rangle = Tr(\mathcal{H} \rho_{GC}) = \left(\frac{\partial(\Omega\beta)}{\partial\beta} \right) = \sum_{p \in \Lambda^*} \varepsilon_p \, n(\varepsilon_p)$$

What happens if $L \to +\infty$

$$n = \lim_{L\to\infty} \frac{\langle \mathcal{N} \rangle}{L^3} = \lim_{L\to\infty} \frac{1}{L^3} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} n(\varepsilon_p)$$

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{e^{\beta(p^2-\mu)}-1} \, dp \quad \text{bosons}$$

$$= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3p \; n(\varepsilon_p) =$$

$$\frac{1}{e^{\beta(p^2-\mu)}+1} \, dp \qquad \text{fermions}$$

When $\mu \in (-\infty, \infty)$ we ~~get~~ get all n in $(0, +\infty)$. fermionic case

What about bosonic case ? We had $\mu < 0$

$\Rightarrow$ the highest possible particle density is obtained for $\mu = 0$

$$n_c(\beta) = \lim_{\mu \to 0} n = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{e^{\beta p^2} - 1} < \infty$$

$\sim \frac{1}{p^2}$ for p small

Thus $\int \dots < \infty$

We have to take

$$\mu = - \frac{1}{\beta L^3 (n - n_c(\beta))} \quad \text{for } n > n_c(\beta)$$

$$\lim_{L\to\infty} \frac{\langle N_0\rangle}{L^3} = \lim_{L\to\infty} \frac{1}{L^3} \frac{1}{e^{\frac{1}{L^3(n-n_c(\beta))}}-1} = \lim_{L\to\infty} \frac{1}{L^3} L^3 (n-n_c(\beta))$$

$$\cancel{1} + \frac{1}{L^3(n-n_c(\beta))} - \cancel{1}$$

$$= (n - n_c(\beta))$$

$\leadsto$ All extra particles go in the zero momentum mode

Macroscopic occupation of the zero momentum mode

Bose Einstein Condensation (BEC)

What happens for $p \neq 0$

Let $p = \left(\frac{2\pi}{L}, 0, 0\right) \rightsquigarrow p^2 = \left(\frac{2\pi}{L}\right)^2$

$$\frac{\langle N_p \rangle}{L^3} = \frac{1}{L^3} \frac{1}{\frac{1}{e^{L^3(n - n_c(\beta))}} e^{\beta\left(\frac{2\pi}{L}\right)^2} - 1} = \frac{L^2}{L^3} \xrightarrow[L \to \infty]{} 0$$

$$\sim \cancel{1} + \frac{c}{L^2} - \cancel{1}$$

Now we would like to go back to the fermionic case

$$x = \beta |p|^2$$

$$n = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{e^{\beta(p^2-\mu)}+1} \, dp$$

$$d^3p = 4\pi \, p^2 dp = \frac{2\pi \sqrt{x}}{\beta^{3/2}} dx$$

$$= \frac{1}{\lambda^3} \frac{2}{\sqrt{\pi}} \int_0^\infty dx \, \frac{\sqrt{x}}{z^{-1}e^{x}+1} = \frac{1}{\lambda^3} f^{+}_{3/2}(z) \rightsquigarrow \frac{\lambda^3}{v} = f^{+}_{3/2}(z)$$

$z = e^{\beta\mu}$, $\lambda = \sqrt{\frac{4\pi}{T}}$ Thermal wave length

What happens $n\lambda^3 = \frac{\lambda^3}{v} \gg 1 \rightsquigarrow \lambda^3 \gg v$ quantum regime

$z \longrightarrow \infty$

where $v = \frac{1}{n}$ volume per particle

$$f^{+}_{\frac{3}{2}}(z) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} dx \, \frac{\sqrt{x}}{z^{-1}e^{x}+1} \, .$$

$y = x - \log z$

$$= \frac{2}{\sqrt{\pi}} \int_{-\log z}^{\infty} dy \, \frac{(y+\log z)^{3/2}}{e^{y}+1}$$

$$= \frac{2}{\sqrt{\pi}} \int_{-\log z}^{\infty} dy \, \frac{2}{3} (y+\log z)^{3/2} \frac{d}{dy}\left(-\frac{1}{1+e^{y}}\right)$$

by parts

$2 \; z^{3/2}$

$$\sim \frac{2}{\sqrt{\pi}} \frac{2}{3} z^{3/2} \int_{-\infty}^{\infty} dy \left(\frac{d}{dy} \frac{-1}{1+e^{y}}\right) \sim \frac{4}{3\sqrt{\pi}} (\log z)^{3/2}$$

$z = e^{\beta\mu}$

So when $z \to \infty$

$$\frac{\lambda^3}{v} \sim \frac{4}{3\sqrt{\pi}} (\log z)^{3/2} \rightsquigarrow \underset{\parallel}{(\log z)^{3/2}} \sim \frac{3\sqrt{\pi}}{4} \frac{\lambda^3}{v} = 6\pi^2 \frac{\beta^{3/2}}{v}$$

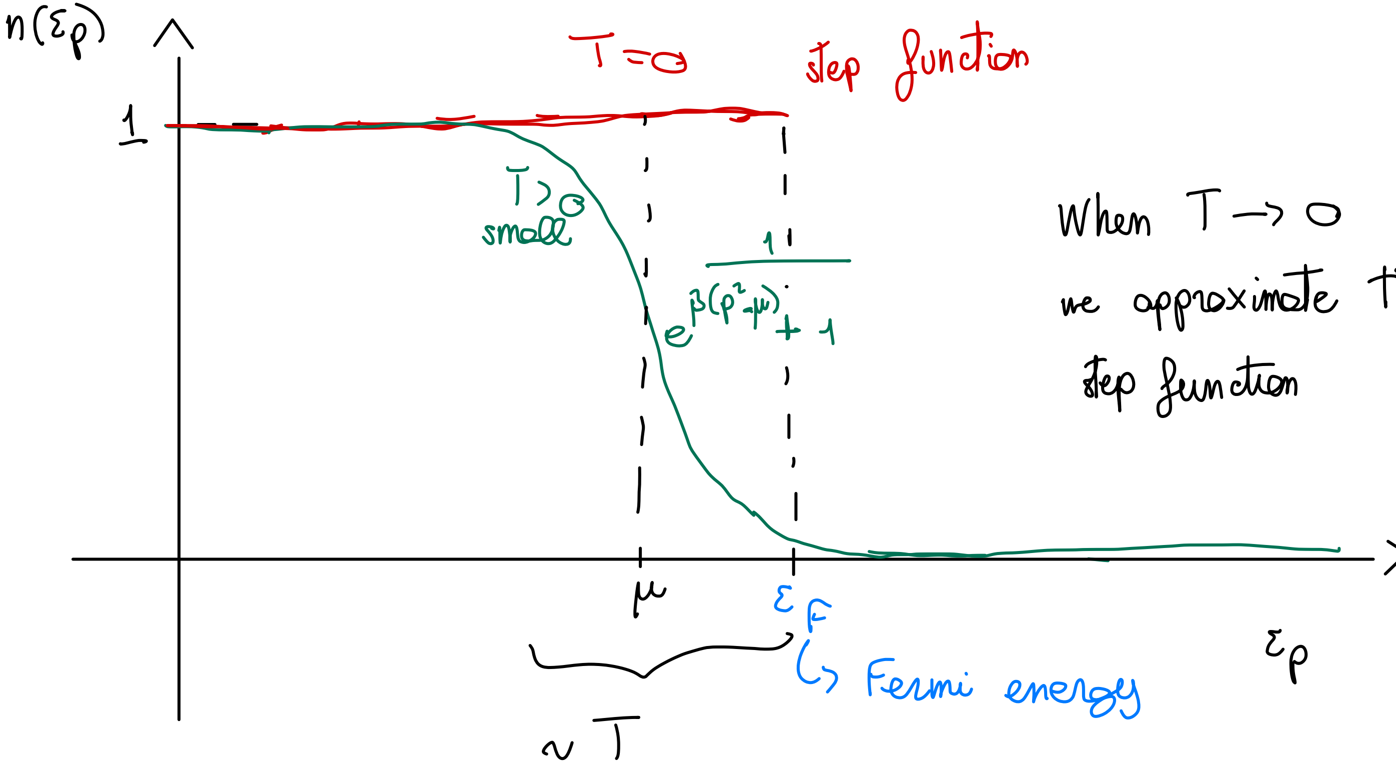
$$(\cancel{\beta}\mu)^{3/2} = \frac{6\pi^2}{v} \cancel{\beta^{3/2}}$$

$$\mu = \left(\frac{6\pi^2}{v}\right)^{2/3} = \varepsilon_F$$

$$N = \sum_{\substack{p: \\ \varepsilon_p < \varepsilon_F}} 1 \rightsquigarrow n = \lim_{L\to\infty} \frac{N}{L^3} \sum_{\varepsilon_p < \varepsilon_F} 1 = \frac{4\pi}{(2\pi)^3} \int_0^{\sqrt{\varepsilon_F}} p^2\, dp = \frac{4}{3} \frac{1}{(2\pi)^3} \pi \varepsilon_F^{3/2}$$

$$\rightsquigarrow \varepsilon_F = (6\pi^2 n)^{2/3} = \left(\frac{6\pi^2}{v}\right)^{2/3}$$

$$\boxed{T = 0}$$



When $T \to 0$ we approximate the step function

$z \to \infty$

$z \to 0 \Rightarrow \frac{\lambda^3}{v} \ll 1$ "classical" regime

One recovers the classical relation for ideal gases

pressure $\leftarrow$ $p v = T$.

$$p = n T \left(1 \pm \frac{1}{2^{5/2}} \frac{\lambda^3}{v} \right) \quad \begin{matrix} + & F \\ - & B \end{matrix}$$

fermions give a positive correction to classical case $\rightsquigarrow$ Pauli

bosons " negative "