Lecture 5

Harmonic Oscillator

and classical limit of QM

HARMONIC OSCILLATOR explicitely solvable model,

prototype model for a particle subject to a confining potential

$V(x) \xrightarrow[|x|\to+\infty]{} +\infty$

Classical harmonic oscillator (1d)

$$H(q,p) = p^2/2m + \frac{1}{2} m \omega^2 q^2 \qquad (q,p) \in \mathbb{R}^2$$

frequency of the oscillation $\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$ spring $F = -kx$

small oscill. of a pendulum $\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta$, $\frac{g}{\ell} = k$

For any initial datum (q_0, p_0) the solution to the Hamilton eqs.

$$\begin{cases} \dot{q} = p/m \\ \dot{p} = -m\omega^2 q = -kq \end{cases} \quad \text{is} \quad \begin{cases} q(t) = q_0 \cos(\omega t) + \frac{p_0}{m\omega}\sin(\omega t) \\ p(t) = -m\omega q_0 \sin(\omega t) + p_0 \cos(\omega t) \end{cases}$$

$\to$ All orbits are BOUNDED; $(q_0, p_0) = 0$ stable equilibrium; $(q_0,p_0) \neq 0$ periodic solution with period $T = 2\pi/\omega$ independent on initial cond.

Step 1 Construction of $H_\omega, D(H_\omega)$ (quantum harmonic oscillator Hamiltonian) s.a. on $L^2(\mathbb{R})$

By quantization procedure

$$\mathring{H}_\omega = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

s.a. extension?

Prop 1 $\mathring{H}_\omega$ has eigenvalues $E_n = \hbar\omega(n+\frac{1}{2})$, $n \in \mathbb{N}$ non degenerate and the corresponding eigenfunctions $\phi_n(x)$ form an ONB in $L^2(\mathbb{R})$

By spect. theorem $H_\omega, D(H_\omega)$ is s.a.

$$D(H_\omega) = \left\{ f \in L^2(\mathbb{R}) : \sum_n E_n \left|\langle \phi_n, f\rangle\right|^2 < \infty \right\}$$

$$\langle H_\omega f, g\rangle = \sum_n E_n \overline{\langle \phi_n, f\rangle} \langle \phi_n, g\rangle$$

Proof of Prop. 1

a) Dimensionless variable $y = \sqrt{\frac{m\omega}{\hbar}}\, x$

$$[y] = \left[\left(\frac{kg \cdot s^{-1}}{\underbrace{kg \cdot m^2 \cdot s^{-1}}_{\hbar}} \right)^{1/2} m \right]$$

Consider the unitary operator $\mathcal{Y}$ in $L^2(\mathbb{R})$

$$(\mathcal{Y}u)(y) = \left(\frac{\hbar}{m\omega}\right)^{1/4} u\left(\sqrt{\frac{\hbar}{m\omega}}\, y\right)$$

$$(\mathcal{Y}^{-1} f)(x) = \left(\frac{m\omega}{\hbar}\right)^{1/4} f\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$$

then $(\mathcal{Y} \mathring{H}_\omega \mathcal{Y}^{-1} f)(y) = \hbar\omega\, (\mathring{H} f)(y)$

$$\mathring{H} = -\frac{1}{2}\frac{d^2}{dy^2} + \frac{1}{2} y^2$$

if λ is an eigenvalue of $\mathring{H}$ the corresponding e.g. of $\mathring{H}_\omega$ is $\hbar\omega\lambda$

b) Solve the eigenvalue problem for

$$\mathcal{H} = -\frac{1}{2}\frac{d^2}{dy^2} + \frac{1}{2}y^2 = \frac{P^2}{2} + \frac{Q^2}{2} \qquad D(\mathcal{H}) = \mathcal{S}(\mathbb{R})$$

$$\Longleftrightarrow \quad \mathcal{H}f = \lambda f, \quad f \in \mathcal{S}(\mathbb{R}), \quad \|f\| = 1$$

<u>TOOL</u> Algebraic method based on the use of the following operators on $\mathcal{S}(\mathbb{R})$

$$A = \frac{1}{\sqrt{2}}(Q + iP) = \frac{1}{\sqrt{2}}\left(\frac{d}{dy} + y\right) \qquad \text{annihilation operator}$$

$$A^* = \frac{1}{\sqrt{2}}(Q - iP) = \frac{1}{\sqrt{2}}\left(-\frac{d}{dy} + y\right) \qquad \text{creation operator}$$

<u>Recall</u> $[Q,P]f = if \quad \forall f \in \mathcal{S}(\mathbb{R})$

LS

Properties $\forall f, g \in \mathcal{S}(\mathbb{R})$

i) $\langle g, Af \rangle = \langle A^* g, f \rangle$ ✓

ii) $[A, A^*] f = f$

$$\underbrace{[A, A^*]}_{} \quad \frac{1}{2}[Q+iP, Q-iP] = \frac{1}{2}\underbrace{[Q, -iP]}_{-i \cdot i \mathbb{1}} + \frac{1}{2}\underbrace{[iP, Q]}_{i(-i)\mathbb{1}} = \mathbb{1}$$

iii) $\mathring{\mathcal{H}} f = A^* A f + \frac{1}{2} f = A A^* f - \frac{1}{2} f$

$$\mathring{\mathcal{H}} = \frac{1}{2}(P^2 + Q^2) \qquad A^* A = \frac{1}{2}(Q - iP)(Q + iP) = \frac{1}{2}(Q^2 + P^2) + \frac{i}{2}\underbrace{[Q, P]}_{-\mathbb{1}}$$

iv) $[\mathring{\mathcal{H}}, A] f = -Af$; $[\mathring{\mathcal{H}}, A^*] f = A^* f$

Lemma 1 Let f_λ solution of $\mathring{\mathcal{H}} f_\lambda = \lambda f_\lambda$, $\|f_\lambda\| = 1$.

then

1) If a solution exists, then $\lambda \geq 1/2$

2) If f_λ is solution with e.g. λ then

$$f_{\lambda'} = \frac{A^* f_\lambda}{\sqrt{\lambda + 1/2}} \quad \text{is solution with e.g.} \quad \lambda' = \lambda + 1$$

3) If f_λ is solution with e.g. λ then

$$f_{\lambda'} = \frac{A f_\lambda}{\sqrt{\lambda - 1/2}} \quad \text{is sol. with e.g.} \quad \lambda' = \lambda - 1$$

Proof of 1)

$$\lambda = \langle f_\lambda, \mathring{\mathcal{H}} f_\lambda \rangle = \langle f_\lambda, A^*A + 1/2\, f_\lambda \rangle = \|A f_\lambda\|^2 + 1/2$$

$$\mathring{\mathcal{H}} = A^*A + 1/2 \qquad \geq 1/2 \quad \square$$

2) If f_λ is solution with eg. λ then

$$f_{\lambda'} = \frac{A^* f_\lambda}{\sqrt{\lambda + 1/2}}$$ is solution with eg. $\lambda' = \lambda + 1$

Proof

$\mathcal{H} = A^*A + 1/2$

$$\mathcal{H}\, A^* f_\lambda = (A^*A + 1/2)\, A^* f_\lambda$$

$\mathcal{H} = AA^* - 1/2$

$$= A^* (AA^* + 1/2) f_\lambda = A^* (\mathcal{H} + 1) f_\lambda$$

$$= A^* (\lambda + 1) f_\lambda$$

$$\| A^* f_\lambda \|^2 = \langle f_\lambda, AA^* f_\lambda \rangle = \langle f_\lambda, (\mathcal{H} + 1/2) f_\lambda \rangle$$

$$= (\lambda + 1/2) \| f_\lambda \|^2$$

$\rightarrow$ $\dfrac{A^* f_\lambda}{\sqrt{\lambda + 1/2}}$ is a normalized eigenv. with eigenv. $(\lambda + 1)$ ✓

Lemma 2 The eigenvalue problem $\mathring{H} f = \lambda f$, $f \in \mathcal{S}(\mathbb{R})$, $\|f\| = 1$ L8

admits a solution if and only if

$$\lambda = n + \tfrac{1}{2} \qquad n = 0, 1, 2 \ldots$$

If $\lambda = n + 1/2$ then the unique solution is

$$f_n(y) = \frac{1}{\sqrt{n!}} \left[(A^*)^n f_0 \right](y)$$

$$f_0(y) = \frac{1}{\pi^{1/4}} e^{-y^2/2}$$

Moreover the system of eigenvectors $\{f_n\}$ is orthon. and complete in $L^2(\mathbb{R})$

Proof of lemma 2. Consider $n=0$.

Since $\lambda = \|Af_\lambda\|^2 + 1/2$, $\lambda = 1/2 \iff Af_0 = 0$

Using $A = \frac{1}{\sqrt{2}}(Q+iP) = \frac{1}{\sqrt{2}}\left(\frac{d}{dy}+y\right)$ $\quad \frac{d}{dy}f_0 + yf_0 = 0$

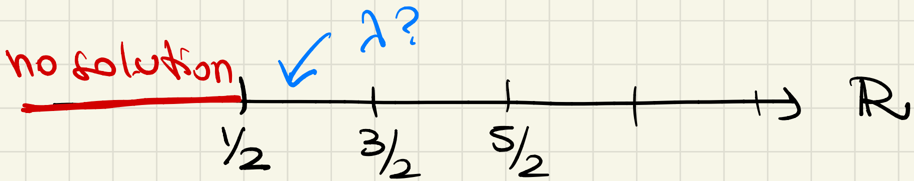
$$\longrightarrow f_0(y) = \frac{e^{-y^2/2}}{\pi^{1/4}}$$

Let us fix $n=1$. By lemma 1, part 1 $\boxed{f_1 = A^* f_0}$ solves the eigenv. probl. with $\lambda_1 = 3/2$. UNIQUENESS?

Assume g_1 solves $\mathring{H} g_1 = \lambda_1 g_1$ with $\lambda_1 = 3/2$. Then by lemma 1, part 3 $Ag_1 = f_0$

$$f_1 \equiv A^* f_0 = A^* A g_1 = (\mathring{H} - 1/2) g_1 = g_1 \Rightarrow f_1 = g_1$$

Repeated application of A^* to the ground state $f_0(y)$ produce eigenf. $f_n(y)$ corresponding to $\lambda_n = n + 1/2$



It remains to show that there are not solutions for $\lambda \neq n + 1/2$

Fix $\lambda \in (1/2, 3/2)$. Assume $\exists\, f_\lambda$ s.t. $\mathring{\mathcal{H}} f_\lambda = \lambda f_\lambda$. Then we can construct $\frac{A f_\lambda}{\sqrt{\lambda - 1/2}}$ well def. since $\lambda > 1/2$

then

$$\mathring{\mathcal{H}} \frac{A f_\lambda}{\sqrt{\lambda - 1/2}} = (AA^* - 1/2) \frac{A f_\lambda}{\sqrt{\lambda - 1/2}} = A(\overbrace{A^*A - 1/2}^{\mathring{\mathcal{H}} - 1}) \frac{f_\lambda}{\sqrt{\lambda - 1/2}} = \overbrace{(\lambda - 1)}^{\in(-1/2,\, 1/2)} \frac{A f_\lambda}{\sqrt{\lambda - 1}}$$

contraddiction!

$\lambda \in (1/2, 3/2)$ not possible!

<u>Next</u> fix $\lambda \in (3/2, 5/2)$, by appl. A you get $f_{\lambda'}$ with $\lambda' \in (1/2, 3/2)$ which is not possible and so on...

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Summarizing

$$\mathring{H} f_n = \lambda_n f_n$$

$$\lambda_n = n + 1/2 \qquad n = 0, 1, 2, \ldots$$

$$f_n = \frac{1}{\sqrt{2^n n!}\, \pi^{1/4}} \underbrace{H_n(y)} e^{-y^2/2}$$

Hermite polynomial of degree n

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad x \in \mathbb{R}$$

$\{f_n\}$ complete in L^2

$\rightarrow$ sect. 7.5 Teta's book

$$\mathring{H}_\omega = \mathcal{Y} \mathring{H} \mathcal{Y}^{-1}$$

$$\mathring{H}_\omega \phi_n = E_n \phi_n$$

$$E_n = \hbar\omega (n + 1/2)$$

$$\phi_n(x) = (\mathcal{Y}^{-1} f_n)(x)$$

$$= \left(\frac{m\omega}{\hbar}\right)^{1/4} f_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$$

$\phi_0(x)$ ground state

$\phi_n(x)$, $n > 0$ excited states

Step 2 Analysis of the spectrum

the spectral measure associated to H_ω is pure point.

$$\sigma(H_\omega) = \sigma_p(H_\omega) = \{ E_n = \hbar\omega(n + 1/2) \quad n = 0, 1, 2, \dots \}$$

RK 1 Note that $E_n = \hbar\omega(n+1/2)$ was the assumption made by Planck to solve the black-body radiation problem

Planck $\Delta E = n \, h\nu = n \overbrace{\frac{h}{2\pi}}^{\hbar} \overbrace{2\pi\nu}^{\omega}$

same ΔE, starting from $E_0 = \frac{\hbar\omega}{2}$

RK2 the fact that $\sigma(H_\omega) \geq 1/2$ can be understood as a consequence of the uncertainty principle

$$\langle \psi, H_\omega \psi\rangle = \frac{1}{2m}\langle \psi, P^2\psi\rangle + \frac{1}{2}m\omega^2\langle \psi, Q^2\psi\rangle$$

$a^2+b^2 \geq 2ab$

$$\geq \omega \sqrt{\langle P^2\rangle_\psi}\sqrt{\langle Q^2\rangle_\psi} \geq \frac{\hbar\omega}{2}$$

$\Delta P\, \Delta Q \geq \hbar/2$

the minimum value $\frac{\hbar\omega}{2}$ is reached if we choose as gaussian wave packet with $x_0 = p_0 = 0$ $\psi_\beta(x) = \left(\frac{\beta}{\pi\hbar}\right)^{1/4} e^{-\frac{\beta}{2\hbar}x^2}$

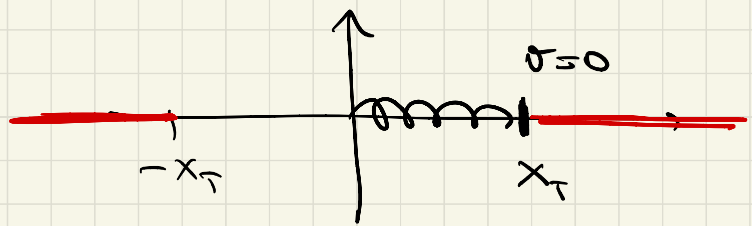
and fix β^* s.t. $\frac{1}{2m}\langle P^2\rangle = \frac{1}{2}m\omega^2\langle Q^2\rangle$

$\Rightarrow$ $\psi_{\beta^*}(x) = \phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$ ground state function

EXERCISE Consider a harmonic oscillator in the first excited state $\phi_1(x)$. Compute the probability to find the particle in the "classically forbidden region"

A classical oscillator with initial energy E will oscillates between the point $-x_T$ and $+x_T$ where

$$\frac{1}{2} m\omega^2 x_T^2 = E$$



If $E = E_1 = \frac{3}{2}\hbar\omega$

$$x_T^2 = \frac{3\hbar}{m\omega}$$

$$P\left(|x| \geq \frac{3\hbar}{m\omega} ; \phi_1\right) = 2\int_{\sqrt{\frac{3\hbar}{m\omega}}}^{+\infty} |\phi_1(x)|^2 dx \qquad ?$$

Using the explicit expression of $\phi_1(x)$: LIS

$$\phi_1(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \frac{1}{\sqrt{2}} H_1\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

$H_1(y) = y$

$$= \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \sqrt{\frac{m\omega}{2\hbar}}\, x \; e^{-\frac{m\omega}{2\hbar}x^2}$$

$$P\left(|x| \geq \frac{3\hbar}{m\omega}; \phi_1\right) = \cancel{2} \int_{\sqrt{\frac{3\hbar}{m\omega}}}^{+\infty} \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \frac{m\omega}{\cancel{2}\hbar} x^2 \, e^{-\frac{m\omega}{\hbar}x^2} dx$$

$y = \sqrt{\frac{m\omega}{\hbar}}\, x$

$$= \frac{1}{\sqrt{\pi}} \int_{\sqrt{3}}^{+\infty} y^2 e^{-y^2} dy \simeq 0.028$$

$\neq 0$

$\rightarrow$ Quantum Mechanical Tunneling

Step 3 Unitary evolution

$$\psi(x,t) = \left(e^{-i\frac{t}{\hbar}H_\omega} \psi_0 \right)(x)$$

$$= \sum_n e^{-i\frac{t}{\hbar}E_n} \langle \phi_n, \psi_0 \rangle \phi_n(x)$$

one can also write this action in terms of an integral operator

Prop 2. Let $f \in L^2(\mathbb{R})$. Then $\forall \varepsilon > 0 \ \exists R_0 > 0$ s.t.

$$\sup_t \int_{|x|>R_0} dx \left| \left(e^{-i\frac{t}{\hbar}H_\omega} f\right)(x) \right|^2 < \varepsilon$$

ie any state is a bound state for the harmonic oscill. Hamiltonian

INTERPRETATION: apart for a small probability the particle is uniformely (in time) confined in a bounded region.

Proof Fix $\varepsilon > 0$. The completeness of $\{\phi_n\}$ implies $\exists\, n_0$ s.t.

$$\underbrace{\| f - \sum_{n=0}^{n_0} \langle \phi_n, f\rangle \phi_n \|}\leq \sqrt{\varepsilon}/2$$

Hence

$$\| \chi_{>R}\, e^{-i\frac{t}{\hbar}H_\omega} f \| \leq \overbrace{\| \chi_{>R}\, e^{-i\frac{t}{\hbar}H_\omega}\Big(f - \sum_{n=0}^{n_0} \langle \phi_n, f\rangle \phi_n\Big)\|}^{\leq \sqrt{\varepsilon}/2}$$

$$+ \| \chi_{>R} \sum_{n=0}^{n_0} e^{-i\frac{t}{\hbar}E_n} \langle \phi_n, f\rangle \phi_n \|$$

$$\leq \frac{\sqrt{\varepsilon}}{2} + \sum_{n=0}^{n_0} |\langle \phi_n, f\rangle|\, \| \chi_{>R} \phi_n \| \leq \sqrt{\varepsilon}$$

Since $\phi_n \in L^2$, $\exists\, R_0 > 0$ s.t.

$$\| \chi_{>R_0} \phi_n \| \leq \frac{\sqrt{\varepsilon}}{2} \frac{1}{(n_0+1) \max_{n \leq n_0} |\langle \phi_n, f\rangle|}$$

The prop. holds for any Hamiltonian with a purely discrete spectrum.

Step 4 Time evolution of a gaussian state

Gaussian state

$$\psi_0(x) = \frac{1}{(\pi\hbar)^{1/4}\sqrt{a_0}}\, e^{-\frac{b_0}{a_0}\frac{(x-q_0)^2}{2\hbar} + i\frac{p_0}{\hbar}(x-q_0)}$$

$\mathrm{Re}\left(\frac{b_0}{a_0}\right) = \frac{1}{|a_0|^2}$

$q_0, p_0 \in \mathbb{R}$, and $a_0, b_0 \in \mathbb{C}$ s.t. $\mathrm{Re}(\bar{a}_0 b_0) = 1$

RK For $a_0 = \sigma\sqrt{\hbar}$, $b_0 = a_0^{-1}$, $\sigma > 0$ $\quad \psi = \frac{1}{\sqrt{\sigma}} \frac{e^{-x^2}}{\pi^{1/4}} e^{i\frac{p_0}{\hbar}x}$

gaussian wave packet

Compute $\quad \langle x\rangle(0) = q_0 \qquad \langle p\rangle(0) = p_0$

$$\Delta x(0) = \sqrt{\frac{\hbar}{2}}\,|a_0| \qquad \Delta p(0) = \sqrt{\frac{\hbar}{2}}\,|b_0|$$

Coherent state : $|a_0|\,|b_0| = 1$ $\qquad \Delta x(0)\Delta p(0) = \frac{\hbar}{2}|a_0||b_0|$

(Eigenstates of A are coherent states)

Prop. the solution of the Schrö d eq. for the harm. oscill with init. datum $\psi_0(x)$ is

$$\psi(x,t) = \frac{1}{(\pi\hbar)^{1/2}\sqrt{a(t)}}\, e^{-\frac{b(t)}{a(t)}\frac{(x-q(t))^2}{2\hbar}+i\frac{p(t)}{\hbar}(x-q(t))}\, e^{\frac{i}{\hbar}S(t)}$$

where $S(t), a(t), b(t), q(t), p(t)$ are functions satisfying

classical Hamilton eq.

$$\begin{cases} \dot q(t) = \dfrac{p(t)}{m} \\ \dot p(t) = -m\omega^2 q(t) \end{cases} \qquad \dot a(t) = i\frac{b(t)}{m} \qquad \dot b(t) = i m\omega^2 a(t)$$

$$\dot S(t) = \frac{p^2(t)}{2m} - \frac{1}{2}m\omega^2 q^2(t) : L(q(t),p(t))$$

classical action computed along the classical motion

with initial condition $S(0)=0$,

$a(0)=a_0,\ b(0)=b_0,\ q(0)=q_0,\ p(0)=p_0.$

Eq. for $a(t)$ & $b(t)$ can be solved

$$a(t) = a_0 \cos(\omega t) + i \frac{b_0}{m\omega} \sin(\omega t)$$

$$b(t) = i m \omega a_0 \sin(\omega t) + b_0 \cos(\omega t)$$

Since $\Delta x(t) = \sqrt{\frac{\hbar}{2}}\, |a(t)|$ $\quad \Delta p(t) = \sqrt{\frac{\hbar}{2}}\, |b(t)|$

mean square deviations of position and momentum are bounded and periodic functions of time.

<u>Rk</u> $a_0 = \frac{1}{\sqrt{m\omega}}$, $b_0 = \sqrt{m\omega}$ we get $a(t) = a_0 e^{i\omega t}$
$b(t) = b_0 e^{i\omega t}$

→ the evolution preserves the form of the state and $\Delta x(t)$, $\Delta p(t)$ <u>do not</u> depend on time.

CLASSICAL LIMIT OF QUANTUM Mechanics

Quantum dynamics
$$\begin{cases} i\hbar \dfrac{\partial \psi}{\partial t} = -\dfrac{\hbar^2}{2m}\Delta\psi + V(x)\psi \\ \psi(0) = \psi_0 \end{cases}$$

$\hbar \to 0$ is a singular limit for the Schrod. eq.

Crucial role in the theory is played by $\hbar$

$$[Q, P]f = i\hbar f \qquad \forall f \in \mathcal{S}(\mathbb{R})$$

Question: if $\hbar$ is sufficiently small (small w.r.t. other typical actions in the system we are considering)

can we neglect the non-commutativity and recover the classical behavior?

We need to consider states ψ depending on $\hbar$

Choice 1 : quasi-classical state

Gaussian state
$$\psi_0(x) = \frac{1}{(\pi\hbar)^{1/4}\sqrt{a_0}}\, e^{-\frac{b_0}{a_0}\frac{(x-q_0)^2}{2\hbar} + i\frac{p_0}{\hbar}(x-q_0)}$$

for $\hbar \to 0$ is very well concentrated around q_0 and p_0

$$\Delta x(0) = \sqrt{\frac{\hbar}{2}}\,|a_0| \qquad \Delta p(0) = \sqrt{\frac{\hbar}{2}}\,|b_0|$$

the evolved state
$$\psi(x,t) = e^{-\frac{it}{\hbar}H}\psi_0 \qquad H = -\frac{\hbar^2}{2m}\Delta + V(x)$$

under suitable assumptions on V (for t not too large)
behaves like a wave packet well concentrated around
the classical trajectory in phase space (for $d \geq 1$)

Prop let $V \in C^3(\mathbb{R})$, $\underbrace{\|V\|_{L^\infty(\mathbb{R})}, \|V'''\|_{L^\infty(\mathbb{R})}}$ - let $S(t), q(t)$, $p(t), a(t), b(t)$ solution of

→ the same result also holds under less restrictive conditions also

Hamilton eq.
$$\begin{cases} \dot{q}(t) = p(t)/m \\ \dot{p}(t) = -V'(q(t)) \end{cases}$$

$$\left.\begin{aligned} \dot{a}(t) &= \frac{i}{m} b(t) \\ \dot{b}(t) &= iV''(q(t))\, a(t) \end{aligned}\right\}$$

linearized flow around the classical solution

classical action
$$\dot{S}(t) = \frac{p^2(t)}{2m} - V(q(t)) = L(q(t), p(t))$$

with initial conditions $q(0) = q_0$, $p(0) = p_0$, $a(0) = a_0$, $b(0) = b_0$, $S(0) = 0$

then there exist $C_V(T) > 0$ (depending on the potential) s.t.

$$\sup_{t \in [0,T]} \| \psi(t) - \phi(t) \|_2 \leq C_V(T) \sqrt{\hbar}$$

$\psi(t)$: quantum sol $e^{-\frac{itH}{\hbar}} \psi_0$

$\phi(t)$: quasi-classical state at time t

Result NOT unif in time!

$C_V(T) \xrightarrow[T \to +\infty]{} +\infty$

Choice 2 : WKB states

Fix $\psi_0 \in L^2(\mathbb{R}^3)$ $\qquad \psi_0(x) = \underbrace{\sqrt{\rho_0(x)}}_{\text{amplitude indep. on } \hbar} \; e^{\frac{i}{\hbar} S_0(x)}$

phase rapidly oscillating in $\hbar$

Write the solution of the Schrö eq. for $H = -\Delta\frac{\hbar^2}{2m} + V(x)$ as

$$\psi(x,t) = \sqrt{\hat\rho(x,t)} \; e^{\frac{i}{\hbar}\hat S(x,t)}$$

one gets

$$\frac{\partial \hat S}{\partial t} + \frac{|\nabla \hat S|^2}{2m} + \overbrace{V \underbrace{- \frac{\hbar^2}{2m}\frac{\Delta\sqrt{\hat\rho}}{\sqrt{\hat\rho}}}_{\text{quantum potential}}}^{W} = 0$$

Hamilton-Jacobi eq. for W

$$\frac{\partial \hat\rho}{\partial t} + \nabla\cdot\left(\frac{\nabla \hat S}{m}\hat\rho\right) = 0$$

transport equation with velocity field $m^{-1}\nabla \hat S$

<u>WKB method's idea</u> for $\hbar \to 0$ the quantum potential becomes negligible and an approximate solution $\eta(t)$ of the Schr. eq. can be written as

$$\eta(x,t) = \sqrt{\rho(x,t)}\, e^{\frac{i}{\hbar} S(x,t)}$$

$$\begin{cases} \dfrac{\partial S}{\partial t} + \dfrac{|\nabla S|^2}{2m} + V = 0 \\[2ex] \dfrac{\partial \rho}{\partial t} + \nabla \cdot \left(\dfrac{\nabla S}{m} \rho \right) = 0 \end{cases}$$

Hamilton-Jacobi eq. for a classical particle

Prop. $\exists\, c(T) > 0$ s.t. $\sup_{t \in [0,T]} \| \psi(t) - \eta(t) \| \leq c(T)\, \hbar$

<u>INTERPRET</u>: the prob. density of the position $\rho(x,t) = |\psi(x,t)|^2$ is localized around the classical trajectory of the particle.

References

Harmonic Oscillator : Sec 7.1 - 7.3 Teta's book

Classical limit of QM :

Appendix A , Teta's book