# Lecture 4

Evolution operator

Free particle

# UNITARY GROUPS

<u>Def.</u> the family of linear operators $U(t)$, $t \in \mathbb{R}$ is a strongly continuous one-parameter group of unitary oper. in $\mathcal{H}$ if

i) $U(t)$ is a unitary op. in $\mathcal{H}$ $\quad \forall t \in \mathbb{R}$ — one-parameter

ii) $U(t+s) = U(t)\,U(s) \quad \forall s,t \in \mathbb{R}$ — group property

iii) $\lim_{t\to 0} \| U(t)f - f \| = 0 \quad \forall f \in \mathcal{H}$

continuity in $t=0$

$\Rightarrow$ continuity $\forall t \in \mathbb{R}$

L2

**Prop.** Let $A, D(A)$ be s.a. on $\mathcal{H}$. Then $U(t) = e^{itA}$, $t \in \mathbb{R}$ is a strongly cont. one-parameter group of unitary operators. Moreover $f \in D(A) \Rightarrow e^{itA} f \in D(A)$ and

$$\rightarrow \frac{d}{dt} e^{itA} f \Big|_{t=0} = \lim_{t \to 0} \frac{1}{t} \left( e^{itA} f - f \right) = iAf$$

**Proof** i) unitarity of $U(t)$ $\forall t \in \mathbb{R}$

$\forall t \rightarrow \phi(\lambda) = e^{it\lambda}$ continuous & bounded

funct. calculus $\longrightarrow U(t) = e^{itA}$ bounded operator

- $U(t)^* = U(-t)$
- $1 = e^{it\lambda} e^{-it\lambda} \Rightarrow \mathbb{1} = U(t)\,U(-t) = U(-t)\,U(t)$

$\Rightarrow U(t)^{-1} = U(-t) = U(t)^* \Leftrightarrow U(t)$ unitary

ii) group prop. $\phi_t(\lambda) = e^{it\lambda}$

$$\phi_{t+s}(\lambda) = \phi_t(\lambda)\,\phi_s(\lambda) \Rightarrow \overbrace{U(t+s)}^{e^{i(t+s)A}} = U(t)\,U(s)$$

iii) $\| e^{itA} f - f \|^2 = \int | e^{it\lambda} - 1 |^2 \, d(\langle E_A(\lambda) f, f\rangle) \xrightarrow[t\to 0]{} 0$ by dominated convergence

• $\| \frac{1}{t} ( e^{iAt} f - f ) - iAf \|^2$

$$= \int \left| \frac{1}{t} \left( e^{it\lambda} - 1 \right) - i\lambda f \right|^2 d(\langle E_A(\lambda) f, f\rangle) \xrightarrow[t\to 0]{} 0 \text{ by dom. conv.}$$

**STONE'S THM** Given a strongly continuous one-parameter group of unitary op. $U(t)$, $t \in \mathbb{R}$, $\exists!$ s.a. operator $A, D(A)$ s.t. $U(t) = e^{itA}$.

## EVOLUTION of a STATE in QM

Let $\mathcal{H}, D(\mathcal{H})$ be the s.a. Hamiltonian of a certain system defined on $\mathcal{H}$. Let $\psi_0 \in \mathcal{H}$ be the initial state. Then the state at time $t > 0$ is def. by

$$\psi(t) = \underbrace{e^{-i\frac{t}{\hbar}\mathcal{H}}}_{\text{PROPAGATOR}} \psi_0$$

- $\psi(t) = e^{-i\frac{t}{\hbar}\mathcal{H}}\psi_0$ is solution to the Cauchy problem

$$\begin{cases} i\hbar \frac{\partial}{\partial t}\psi(t) = \mathcal{H}\psi(t) \\ \psi(0) = \psi_0 \in D(\mathcal{H}) \end{cases}$$

for a fixed $T>0$ $\exists!$

$\psi(t) \in C^0([0,T], D(\mathcal{H})) \cap C^1([0,T], \mathcal{H})$

- $\exists$ of solution $\iff$ $\mathcal{H}, D(\mathcal{H})$ is self-adjoint

- uniqueness of the sol. follows from $\|\psi_t\|_2 = \|\psi_0\|_2 \ \forall t$

Assume $\psi_1(t), \psi_2(t)$ sol. with initial state $\psi_0$

$\rightarrow$ $\psi_1(t) - \psi_2(t)$ is sol. with initial state 0

$$\|\psi_1(t) - \psi_2(t)\|_2 = \|0\|_2 = 0 \implies \psi_1(t) = \psi_2(t)$$

**Exercise** $H_D = -\Delta$ on $L^2((0,\pi))$

$$D(H_D) = \left\{ f : \sum_n n^4 |\langle \phi_n, f\rangle|^2 < \infty \right\}$$

$$\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx \quad n = 1, 2, \ldots$$

$\hookrightarrow \lambda_n = n^2$

**Verify**

$$(e^{-itH_D} f)(x) = \sum_n \underbrace{e^{-itn^2}}_{\text{evolution of } \phi_n(x)} \langle \phi_n, f\rangle \phi_n(x)$$

# FREE PARTICLE

Classical case: $H_0^d = p^2/2m$ $\leftarrow$ trivial on $\mathbb{R}^d$

## Step 1. construction of $H$, $D(H)$ as s.a. op. on $L^2(\mathbb{R}^d)$

By Weyl quantization $p_j \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial x_j}$

$$\dot{H}_0 = -\frac{\hbar^2}{2m}\Delta \quad , \quad D(\dot{H}_0) = \mathcal{S}(\mathbb{R}^d)$$

We use that $\dot{H}_0$ is a multiplication operator in Fourier

Is it s.a.? If not how to find the self-adjoint ext?

$$(\hat{H}_0 \hat{f})(k) := (\underbrace{\mathcal{F} H_0 \mathcal{F}^{-1}}_{:=\hat{H}_0} \hat{f})(k) = \frac{\hbar^2 k^2}{2m} \hat{f}(k)$$

$\lambda$

Hence

$\int d\lambda\, \lambda^2 |f(\lambda)|^2 < \infty$

$$D(\hat{H}_0) = \{ f \in L^2(\mathbb{R}^d) : \int dk\, |k^2 \hat{f}(k)|^2 < \infty$$

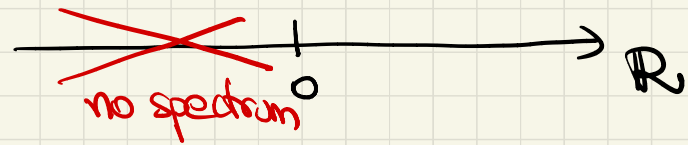
We conclude

$$\boxed{\begin{aligned} (H_0 f)(x) &= (\underbrace{\mathcal{F}^{-1} \hat{H}_0 \mathcal{F}} f)(x) = \int \frac{d^d k}{(2\pi)^{d/2}} e^{ik\cdot x} \frac{\hbar^2 k^2}{2m} \hat{f}(k) \\ D(H_0) &= \{ f \in L^2(\mathbb{R}^d) : \int dk\, |k^2 \hat{f}(k)|^2 < \infty \} := H^2(\mathbb{R}^d) \end{aligned}}$$

## STEP 2 - Analysis of the spectrum of $\mathcal{H}, D(\mathcal{H})$

- $\mathcal{H}_0$ is s.a. $\Longrightarrow$ $\sigma(\mathcal{H}_0) \subseteq \mathbb{R}$

- The resolvent in Fourier space is




$$\phi_z(\lambda) = \frac{1}{\lambda - z} \longrightarrow \frac{1}{\frac{\hbar^2 k^2}{2m} - z}$$

it is well defined as a bounded operator $\Longleftrightarrow z < 0$

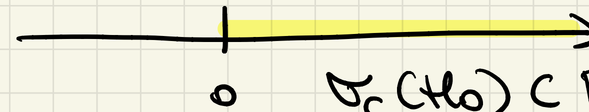
- $\mathcal{H}_0$ has no eigenvalues. Assume that for $\lambda \in \mathbb{R}^+$,

$\exists\ \phi_\lambda \in D(\mathcal{H}_0)$ s.t. $\hat{\mathcal{H}}_0 \hat{\phi}_\lambda = \lambda \hat{\phi}_\lambda$

then $0 = \|(\hat{\mathcal{H}}_0 - \lambda)\hat{\phi}_\lambda\|^2 = \int \left(\frac{\hbar^2 k^2}{2m} - \lambda\right)^2 |\hat{\phi}_\lambda(k)|^2 \, dk$

$\Longleftrightarrow \hat{\phi}_\lambda = 0$ a.e. $\Longrightarrow$ $\lambda$ is not an eigenvalue

How to characterize $\sigma_c(H_0)$?



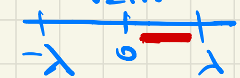

$\sigma_c(H_0) \subset \mathbb{R}^+$

Spectral family

$$(\widehat{E_0(\lambda) f})(k) = \chi_{(-\infty,\lambda]}\left(\frac{\hbar^2 k^2}{2m}\right)\hat{f}(k) = \begin{cases} \hat{f}(k) & \frac{\hbar^2 k^2}{2m} \le \lambda \\ 0 & \text{otherwise} \end{cases}$$

$d=1$

$-\lambda \le \frac{\hbar k}{\sqrt{2m}} \le \lambda$




$$\langle \hat{E}_0(\lambda) f, g \rangle = \int \chi_{(-\infty,\lambda]}\left(\frac{\hbar^2 k^2}{2m}\right) \overline{\hat{f}(k)}\, \hat{g}(k)\, dk$$

fix $\lambda > 0$

$$= \chi_{(0,\infty)}(\lambda) \int_0^{+\infty} \chi_{(0,\lambda]}\left(\frac{\hbar^2 k^2}{2m}\right) \left( \overline{\hat{f}(k)}\, \hat{g}(k) + \overline{\hat{f}(-k)}\, \hat{g}(-k) \right)$$

$$= \chi_{(0,\infty)}(\lambda) \int_0^{\sqrt{2m\lambda}/\hbar} \left( \overline{\hat{f}(k)}\, \hat{g}(k) + \overline{\hat{f}(-k)}\, \hat{g}(-k) \right) dk$$

the distr. function $\lambda \longrightarrow \langle E_0(\lambda) f, f \rangle$ is abs. continuous w.r.t. Lebesgue

$\forall f \in L^2(\mathbb{R}) \longrightarrow m_0^f$ is abs. continuous

$$m_0^f((a,b]) = \langle E_0(b) f, f \rangle - \langle E_0(a) f, f \rangle$$

$$= \int_a^b \frac{d}{d\lambda} \langle E_0(\lambda) f, f \rangle \, d\lambda$$

<u>Exercise</u>: compute $\frac{d}{d\lambda} \langle E_0(\lambda) f, f \rangle$

$$\sigma(H_0) = \sigma_{ac}(H_0) = [0, +\infty)$$

## Step 3 UNITARY EVOLUTION

free propagator

$$U_0(t) = e^{-i\frac{t}{\hbar}H_0}$$

$$(U_0(t)f)(x) = \frac{1}{(2\pi)}\int e^{ik\cdot x}\, \overbrace{e^{-i\frac{t}{\hbar}\frac{\hbar^2 k^2}{2m}}}^{e^{-\frac{i\hbar t}{2m}k^2}}\, \hat{f}(k)\, dk$$

[ **Check** Same result by using func. cal.

$$\langle g, U_0(t)f\rangle = \int_0^{+\infty} e^{-i\frac{t}{\hbar}\lambda}\, d(\langle E_0(\lambda)g, f\rangle)$$
]

**Next step** expression of $U_0(t)$ as an integral operator

**Prop.** Let $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ then

$$\left(e^{-i\frac{t}{\hbar}H_0} f\right)(x) = \left(\frac{m}{2\pi i \hbar t}\right)^{1/2} \int f(y)\, e^{i\frac{m}{2\hbar t}(x-y)^2}\, dy \qquad (1)$$

Moreover, if $f \in L^2(\mathbb{R})$

$$\left(e^{-i\frac{t}{\hbar}H_0} f\right)(x) = \left(\frac{m}{2\pi i \hbar t}\right)^{1/2} \lim_{n\to+\infty} \int_{|y|<n} f(y)\, e^{i\frac{m}{2\hbar t}(x-y)^2}\, dy \qquad (2)$$

where the limit is in the $L^2$-sense.

**Rk.** From (1) one obtains

$$\sup_{x\in\mathbb{R}^d} \left|\left(e^{-i\frac{t}{\hbar}H_0} f\right)(x)\right| \leq \frac{m}{2\pi\hbar} \frac{1}{|t|^{1/2}} \|f\|_1$$

DISPERSIVE character of the free evolution

**Prop.** Let $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ then

$$\left(e^{-i\frac{t}{\hbar}H_0} f\right)(x) = \left(\frac{m}{2\pi i \hbar t}\right)^{1/2} \int f(y)\, e^{i\frac{m}{2\hbar t}(x-y)^2}\, dy \qquad (1)$$

Moreover, if $f \in L^2(\mathbb{R})$

$$\left(e^{-i\frac{t}{\hbar}H_0} f\right)(x) = \left(\frac{m}{2\pi i \hbar t}\right)^{1/2} \lim_{n\to+\infty} \int_{|y|<n} f(y)\, e^{i\frac{m}{2\hbar t}(x-y)^2}\, dy \qquad (2)$$

Show that (1) $\Rightarrow$ (2). For $f \in L^2(\mathbb{R})$, take $f_n := \chi_{<n} f$ then $f_n \in L^2 \cap L^1$ and (1) holds for $f_n$ with $\int dy$ $\to \int_{|y|<n}$.

Now $\| e^{-i\frac{t}{\hbar}H} f - e^{-i\frac{t}{\hbar}H} f_n \| \underset{\substack{\uparrow \\ \text{unitarity of the group}}}{=} \| f - f_n \| \xrightarrow[n\to+\infty]{} 0$

II.5

**Proof 1)** $\forall g, f \in \mathcal{S}(\mathbb{R})$ → by density argument one extend for $f \in L^2 \cap L^1$

$$\langle g, e^{-i\frac{t}{\hbar}H_0} f\rangle = \int \overline{\hat{g}(k)}\, e^{-i\boxed{\frac{\hbar t}{2m}}k^2} \hat{f}(k)\, dk \qquad \frac{\hbar t}{2m} := \tau$$

$$= \lim_{\varepsilon\to 0} \int \overline{\hat{g}(k)}\, \hat{f}(k)\, e^{-(\varepsilon + i\tau)k^2} dk$$

$$= \lim_{\varepsilon\to 0} \int \overline{g(x)} f(y) \int e^{-\underbrace{(\varepsilon+i\tau)}_{a}k^2 + ik\underbrace{(x-y)}_{\lambda}} dk\, dx\, dy$$

Use $\displaystyle\int_{-\infty}^{+\infty} dt\, e^{-at^2+i\lambda t} = \sqrt{\frac{\pi}{|a|}}\, e^{-\lambda^2/4a - \frac{i}{2}\phi}$ $\quad a = |a| e^{i\phi}$, $\lambda \in \mathbb{R}$

$$= \lim_{\varepsilon\to 0} \frac{\sqrt{\pi}}{(2\pi)} \int \overline{g(x)} f(y) \frac{1}{\underbrace{(t^2+\varepsilon^2)^{1/4}}_{|a|^{1/2}}}\, e^{-\frac{(x-y)^2}{4(it+\varepsilon)}}\, e^{-\frac{i}{2}\tan^{-1}(\tau/\varepsilon)} dx\, dy$$

$$\overset{\text{by dom. conv.}}{=} \frac{1}{2\sqrt{\pi\tau}} \int \overline{g(x)} f(y)\, e^{-\frac{(x-y)^2}{4i\tau}}\, e^{-\frac{i}{2}\frac{\pi}{2}} \qquad e^{-\frac{i}{2}\frac{\pi}{2}} = (-i)^{1/2} \qquad \blacksquare$$

## STEP 4 APPLICATION

**Prop.** Asymptotic behavior. For any $\phi \in L^2(\mathbb{R})$ we have, let

$$\phi_t^a(x) = \left(\frac{m}{i\hbar t}\right)^{1/2} e^{\frac{im}{2\hbar t}x^2}\hat{\phi}\left(\frac{mx}{\hbar t}\right), \quad \lim_{t\to\pm\infty} \left\| e^{-i\frac{t}{\hbar}H_0}\phi - \phi_t^a \right\| = 0$$

**Proof** Fix $\phi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

$$\left(e^{-i\frac{t}{\hbar}H_0}\phi\right)(x) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2}\int \phi(y)\, e^{i\frac{m}{2\hbar t}(x-y)^2}\, dy$$

$$= \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} e^{i\frac{m}{2\hbar t}x^2}\int \phi(y)\, e^{-i\frac{m}{\hbar t}xy}\, e^{i\frac{m}{2\hbar t}y^2}\, dy$$

$$\hat{\phi}\left(\frac{mx}{\hbar t}\right)$$

$$:= \phi_t^a(x) + R_t(x)$$

GOAL Show $\|R_t\| \xrightarrow[t\to\pm\infty]{} 0$

$$R_t(x) = \left(\frac{m}{i\hbar t}\right)^{1/2} e^{i\frac{m}{2\hbar t}x^2} \underbrace{\frac{1}{2\pi}\int \underbrace{\phi(y)\left(e^{i\frac{m}{2\hbar t}y^2}-1\right)}_{h(y)} e^{-i\frac{m}{\hbar t}xy}\,dy}_{\hat{h}\left(\frac{mx}{\hbar t}\right)}$$

$$\|R_t\|^2 = \frac{m}{\hbar|t|}\int dx\,\left|\hat{h}\left(\frac{mx}{\hbar t}\right)\right|^2$$

$z = \frac{mx}{\hbar t}$

$$\leq \int dz\,|\hat{h}(z)|^2 = \|h\|^2$$

$$= \int |\phi(y)|^2 \left|e^{i\frac{m}{2\hbar t}y^2}-1\right|^2 dy \xrightarrow[t\to\pm\infty]{} 0$$

by dom. convergence

For $\phi \in L^2$ the result obtained by a density argument.

# SCATTERING

<u>Prop</u> Any state is a scattering state (o diffusion state) w.r.t. the free dynamics ie $\forall$ fixed $R>0$, $\forall \phi \in L^2(\mathbb{R})$ $\phi_t = e^{-i\frac{t}{\hbar}H_0}\phi$ is s.t.

$$\lim_{t\to\pm\infty} \int_{|x|<R} |\phi_t(x)|^2 dx = 0$$

Hence

$$\mathbb{P}\left(x\in(-R,R);\, e^{-i\frac{t}{\hbar}H_0}\phi\right) = \int_{-R}^{R} |\phi_t(x)|^2 dx \xrightarrow[t\to\pm\infty]{} 0$$

<u>INTERPRET</u> Irresp. on the initial condition and on the choice of R after a suff. long time the particle exits $[-R,R]$ (or the $d$-dim. ball of rad. R in higher dimensions)

Proof $(d=1)$

$$\phi_t^a(x) = \left(\frac{m}{i\hbar t}\right)^{1/2} e^{\frac{imx^2}{2\hbar t}} \hat{\phi}\left(\frac{mx}{\hbar t}\right)$$

$$\| \chi_{<R}\, e^{-i\frac{t}{\hbar}H_0} \phi \|$$

$$\leq \| \chi_{<R} ( e^{-i\frac{t}{\hbar}H_0}\phi - \phi_t^a ) \| + \| \chi_{<R}\, \phi_t^a \|$$

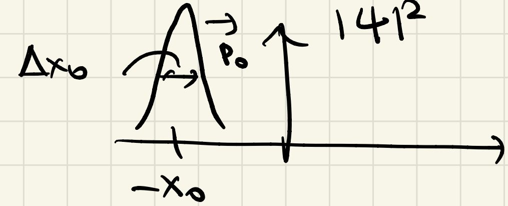
$$\leq \underbrace{\| ( e^{-i\frac{t}{\hbar}H_0}\phi - \phi_t^a ) \|}_{\xrightarrow{t\to+\infty} 0} + \underbrace{\int_{-R}^{R} \frac{m}{\hbar|t|} \left|\hat{\phi}\left(\frac{mx}{\hbar t}\right)\right|^2 dx}_{= \int_{-\frac{mR}{\hbar t}}^{\frac{mR}{\hbar t}} |\hat{\phi}(y)|^2 dy \xrightarrow{t\to+\infty} 0}$$

due to the asymp. behavior result

$y = \frac{mx}{\hbar t}$

$\square$

# FREE EVOLUTION of a wave packet




Let $f \in \mathcal{S}(\mathbb{R})$, real, even, $\|f\| = 1$

$$\Delta x_0 \Delta p_0 \geq \frac{\hbar}{2}$$

$$\psi_0^{x_0, p_0, \sigma} = \frac{1}{\sqrt{\sigma}} f\left(\frac{x + x_0}{\sigma}\right) e^{i \frac{p_0}{\hbar} x} \quad \text{wave packet}$$

for $x_0, p_0, \sigma > 0$ given parameters

<u>Check</u> $\quad \langle x \rangle(0) = -x_0$ (f is even) $\qquad \langle p \rangle(0) = p_0$

$$\Delta x(0)^2 = \sigma^2 \|\hat{f}'\|_2^2 \qquad \Delta p(0)^2 = \frac{\hbar^2}{\sigma^2} \|f'\|_2^2$$

$$\Longrightarrow \quad \Delta x(0)\, \Delta p(0) = \hbar \, \|f'\| \, \|\hat{f}'\|$$

<u>Rk</u> Minimal uncertainty is reached $\|f'\| = \|\hat{f}'\| = 1/\sqrt{2}$. this is the case $f(x) = e^{-x^2/2}/\pi^{1/4} \longleftrightarrow \hat{f}(k) = e^{-k^2/2}/\pi^{1/4}$ GAUSSIAN WAVE PACKET

Time evol $\quad \psi_t = e^{-i\frac{t}{\hbar}H_0} \psi_0^{x_0, p_0, a}$

- $\langle p \rangle(t) = \int dx\, \overline{\psi_t(x)} \frac{\hbar}{i} \frac{d}{dx} \psi_t(x) = \hbar \int dk\, k\, |\hat{\psi}_t(k)|^2 = \langle p \rangle(0) = p_0$

$$|\hat{\psi}_t(k)|^2 = |\hat{\psi}_0(k)|^2$$

$$\hat{\psi}_t(k) = e^{-i\frac{\hbar t k^2}{2m}} \hat{\psi}_0(k)$$

the momentum is a constant of motion for the free particle evol.

$\langle \Delta p \rangle(t) = \langle \Delta p \rangle(0)$

- $\langle x \rangle(t) = \int dk\, \overline{\hat{\psi}_t(k)}\, i \frac{d}{dk} \hat{\psi}_t(k)$

$$= \int dk\, \overline{\hat{\psi}_0(k)} \left( \frac{\hbar t}{m} \hat{\psi}_0(k) + i \frac{d}{dk} \hat{\psi}_0(k) \right)$$

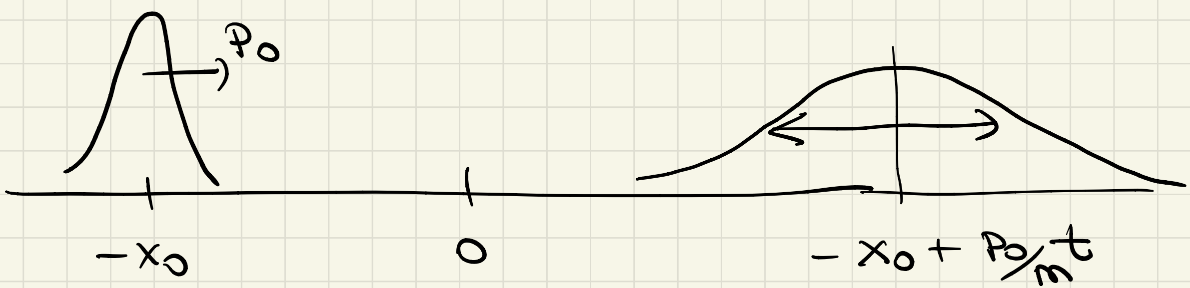
$$= \frac{p_0}{m} t - x_0$$

the mean value is moving with uniform velocity $p_0/m$ towards the right, starting from $-x_0$

For a gaussian wave packet an explicit comput. gives:

$$\Delta x(t)^2 = \underbrace{\left(\frac{t}{m}\right)^2 \Delta p(0)^2}_{} + \Delta x(0)$$

increase in time: the wave packet is going to diffuse as time goes on



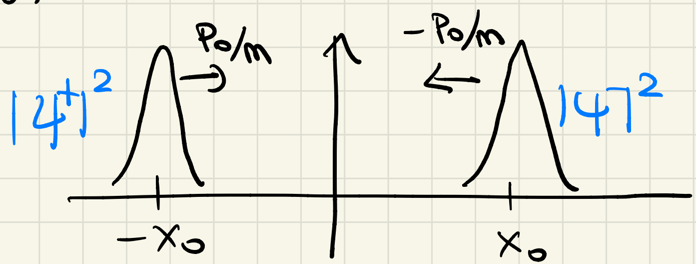

→ we lost information on the position of the particle.

# INTERFERENCE of Two wave packets

Consider the <u>one particle state</u>

$$\psi_0 = c\left[\overbrace{\frac{1}{\sqrt{\sigma}} f\left(\frac{x+x_0}{\sigma}\right) e^{i\frac{p_0 x}{\hbar}}}^{\psi^+(x)} + \overbrace{\frac{1}{\sqrt{\sigma}} f\left(\frac{x-x_0}{\sigma}\right) e^{-i\frac{p_0 x}{\hbar}}}^{\psi^-(x)}\right]$$

normalization constant




$\frac{\sigma}{x_0} \ll 1$ well separated bumps

<u>Evolved state</u>

$$\psi_t(x) = c\left[e^{-i\frac{t}{\hbar}E_0}\psi^+(x) + e^{-i\frac{t}{\hbar}E_0}\psi^-(x)\right]$$

<u>Question</u> what happens at time $t$ : $\frac{p_0}{m}t = x_0$, ie when the centers of the two bumps reach the origin.

Compute $\psi_t(x)\Big|_{t^* = m x_0 / p_0}$. In the gaussian case one finds

$$|\psi_{t^*}(x)|^2 = \underbrace{\frac{4c^2}{\sqrt{\pi}\,\sigma\sqrt{1+\alpha^2}}\, e^{-\frac{x^2}{\sigma^2(1+\alpha^2)}}}_{\text{gaussian bump with variance } \sigma\sqrt{1+\alpha^2} \gg \sigma \text{ (expected)}} \underbrace{\cos^2\left(\frac{p_0}{\hbar}x\right)}_{\text{modulation factor which oscillates over space}}$$

with $\alpha = \frac{\hbar x_0}{\sigma^2 p_0} \gg 1$

gaussian bump with variance $\sigma\sqrt{1+\alpha^2} \gg \sigma$ (expected)

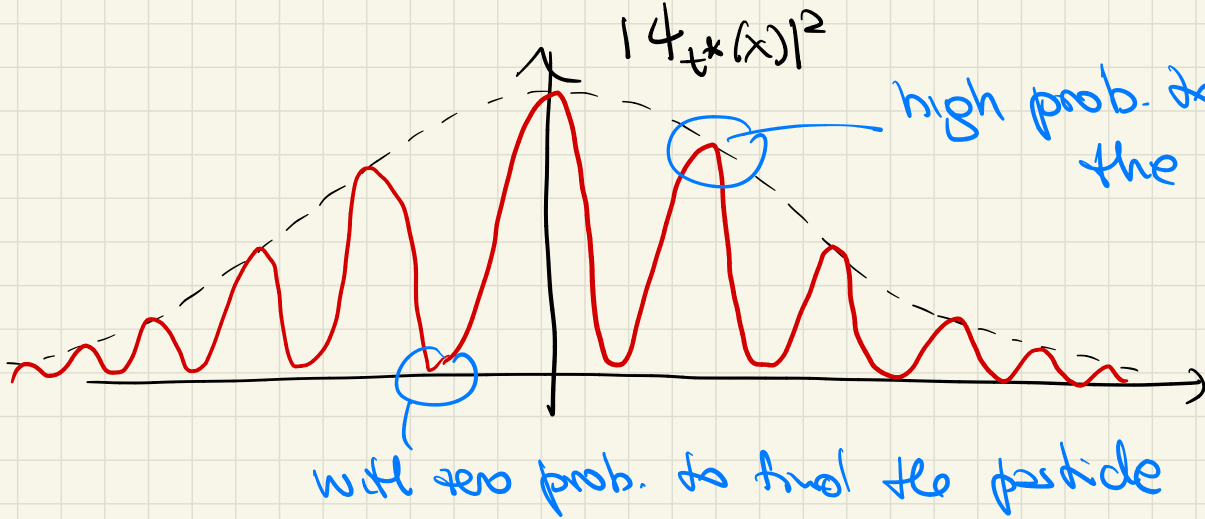
modulation factor which oscillates over space

<u>RK</u> zero of $\cos^2(\cdot) \iff x_k^0 = (2k+1)\frac{\pi}{2}\frac{\hbar}{p_0}$ $\quad k \in \mathbb{Z}$

# zeros in $|x| < \sigma\sqrt{1+\alpha^2} \iff |2k+1|\frac{\pi}{2}\frac{\hbar}{p_0} < \sigma\sqrt{1+\alpha^2}$

$\iff |2k+1| < \frac{2}{\pi}\frac{p_0}{\hbar}\sigma\,\frac{\hbar x_0}{\sigma^2 p_0}\sqrt{1+\alpha^{-2}} \sim \frac{x_0}{\sigma} \gg 1$

lots of zeros in the area where $|\psi|^2 \neq 0$




When the 2 packets overlap we have an <u>INTERFERENCE</u> figure – with no classical <u>counterpart</u>!

$\longrightarrow$ TWO SLIT <u>EXPERIMENT</u>

<u>Rk</u> you <u>cannot</u> interpret $\psi_0(x)$ with two bumps by saying that the particle is to the right with some probability or to the left with complementary prob.

# References

Propagator of the quantum evolution

Sec. 4.9 Teta's book

Free particle

Sec 6.1 - 6.4 Teta's book