Lecture 5

Spectral theorem for
self-adjoint operators

Unitary groups

STIELTJES MEASURE

let $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

DISTRIBUTION FUNCTION $F: \mathbb{R} \to \mathbb{R}$ non decreasing, continuous from the right, s.t.

$$\lim_{\lambda \to -\infty} F(\lambda) = 0 \quad , \quad \lim_{\lambda \to +\infty} F(\lambda) = \alpha \qquad 0 < \alpha < \infty$$

Given F, $\exists !$ finite Borel measure m^F s.t.

$$\to \quad m^F((a,b]) = F(b) - F(a) \qquad a, b \in \mathbb{R} \quad a < b$$

Check
i) $m^F(\{a\}) = F(a) - \lim_{\lambda \to a^-} F(\lambda)$
ii) $m^F(\mathbb{R}) = \alpha < \infty$

Examples A) $F(\lambda) = \chi_{[0,\infty)}(\lambda) \leftarrow$

$$m^F = \delta_0 \qquad \delta_0(B) = \begin{cases} 1 & 0 \in B \\ 0 & \text{otherwise} \end{cases}$$

pure point measure

Check $F(\lambda) = 1$ if $\lambda \in [0, \infty)$ otherwise zero

$$m^F((a,b]) = 1 \qquad \text{if } \underline{b > 0 \;\; a < 0}$$

$F(b) - F(a)$

$$m^F(\{0\}) = F(0) - \lim_{x \to 0^-} F(x) = 1$$

let $a_i \in \mathbb{R}$, $f_i > 0$ $\quad i = 1, \dots, n$

s.t. $\sum_i f_i < \infty$

$$F(x) = \sum_{i \,:\, a_i < x} f_i$$

piecewise constant with discontinuities in a_i

$$f_i = F(a_i) - \lim_{x \to a_i^-} F(x)$$

$$\boxed{m^F(B) = \sum_{i \,:\, a_i \in B} f_i}$$

B) let F an abs. continuous distr. function then

$$m^F((a,b]) = F(b) - F(a) = \int_a^b d\lambda\, \underbrace{F'(\lambda)}_{\text{well defined}}$$

and clearly

$$m^F(B) = \int_B d\lambda\, F'(\lambda) \quad \Leftarrow \text{ Stieltjes measure is abs. cont. w.r.t. Lebesgue}$$

__RK__ if F is continuous but not __abs cont__ (ex Cantor function) $\longrightarrow$ m^F is singular continuous

$\longrightarrow$ sect. 7.4 Reed-Simon I

INTEGRALS Given F, m^F one can define the Lebesgue integral w.r.t. m^F. For Φ complex-valued integrable function we denote its Lebesgue int. by

$$\int_B \phi(\lambda)\, \underline{\underline{dF(\lambda)}}$$

lebesgue meas. associated to m^F

Ex. A) $F(x) = \sum_{i:\, a_i < x} f_i \quad \longrightarrow \quad \int_B \phi(\lambda)\, dF(\lambda) = \sum_{i:\, a_i \in B} \phi(a_i)\, f_i$

Ex. B) F abs. cont. $\quad \int_B \phi(\lambda)\, dF(\lambda) = \int_B \phi(\lambda)\, F'(\lambda)\, d\lambda$

→ monotone converg., dominated conv., Fatou's lemma, Fubini's thm. hold

SPECTRAL FAMILY (resolution of the identity)

Def. $\{E(\lambda)\}$, $\lambda \in \mathbb{R}$ family of orthogonal projectors

s.t. $\forall f \in \mathcal{H}$

(i) $\lim_{\lambda \to -\infty} E(\lambda) f = 0$ $\quad\quad$ $\lim_{\lambda \to +\infty} E(\lambda) f = f$

(ii) $\langle E(\lambda) f, f \rangle \leq \langle E(\mu) f, f \rangle$ if $\lambda \leq \mu$

monotonicity prop.

(iii) $\lim_{\varepsilon \to 0^+} E(\lambda + \varepsilon) f = E(\lambda) f$ $\quad$ continuity from the right

<u>Ex.</u> A'

$$\{\phi_n\}_{n\in\mathbb{N}} \text{ ONB in } \mathcal{H}$$

$$E(\lambda) = \sum_{n:\, n\le\lambda} \langle \phi_n, \cdot \rangle \phi_n \leftarrow \text{projector!}$$

<u>Show</u> $E(\lambda)$ is a spectral family

i) evident

$\mu > \lambda$

ii) $$\langle E(\lambda)f, f\rangle = \sum_{n:\, n\le\lambda} |\langle\phi_n, f\rangle|^2 \le \sum_{n:\, n\le\mu} |\langle\phi_n, f\rangle|^2$$

iii) $$\| E(\lambda+\varepsilon)f - E(\lambda)f \|^2 = \sum_{n:\, \lambda\le n\le\lambda+\varepsilon} |\langle\phi_n, f\rangle|^2 \xrightarrow[\varepsilon\to 0^+]{} 0$$

∎

<u>Ex</u>. B' let us consider the following mult. op. on $L^2(\mathbb{R})$

$$(E(\lambda)f)(x) = \chi_{(-\infty,\lambda]}(x)\, f(x) \qquad \forall f \in L^2(\mathbb{R})$$

<u>Check</u> $E(\lambda)$ is a spectral family

<u>Proof</u> • $\forall \lambda \in \mathbb{R}$, $E(\lambda)$ is linear, bounded, symm.

$E^2 = E \iff$ orthogonal projec

i) ✓

ii) $$\langle E(\lambda)f, f\rangle = \int_{-\infty}^{\lambda} |f(x)|^2\, dx \le \int_{-\infty}^{\mu} |f(x)|^2\, dx \quad \forall \lambda \le \mu$$

iii) $$\|(E(\lambda+\varepsilon) - E(\lambda))f\|^2 = \int_{\lambda}^{\lambda+\varepsilon} |f(x)|^2\, dx \xrightarrow[\varepsilon \to 0^+]{} 0$$

The SPECTRAL MEASURE

Given $f \in \mathcal{H}$, $\{E(\lambda)\}$ spectral family on $\mathcal{H}$ then

$$F^f(\lambda) = \langle E(\lambda)f, f\rangle$$ is a distribution function

Def. For any $f \in \mathcal{H}$, the measure m^f in $\mathbb{R}$ generated by the distr. function $F^f(\lambda)$ is called spectral measure associated to f and $\{E(\lambda)\}$

we can also introduce a complex spectral measure

$$\forall f,g\in\mathcal{H}, \quad \langle E(\lambda)f,g\rangle = \frac{1}{4}\overbrace{\langle E(\lambda)(f+g), f+g\rangle}^{m^{f+g}}$$

Polarization identity

$$\left.\begin{aligned} &-\frac{1}{4}\langle E(\lambda)(f-g), f-g\rangle \\ &+\frac{i}{4}\langle E(\lambda)(f-ig), f-ig\rangle \\ &-\frac{i}{4}\underbrace{\langle E(\lambda)(f+ig), f+ig\rangle}_{m^{f+ig}} \end{aligned}\right.$$

Def For any $f,g\in\mathcal{H}$

$$m^{f,g} := \frac{1}{4}m^{f+g} - \frac{1}{4}m^{f-g} + \frac{i}{4}m^{f-ig} - \frac{i}{4}m^{f+ig}$$

complex spectral measure associated to f,g and $\{E(\lambda)\}$

Ex A' $\quad E(\lambda) = \sum_{n:\, n\leq\lambda} \langle \phi_n, \cdot \rangle \phi_n$, $\{\phi_n\}$ ONB on $\mathcal{H}$ 11

$$F^f(\lambda) = \langle E(\lambda) f, f \rangle = \sum_{n:\, n\leq\lambda} |\langle \phi_n, f \rangle|^2$$

$$m^f(B) = \sum_{n:\, n\in B} |\langle \phi_n, f \rangle|^2$$

$$\int_B \varphi(\lambda)\, d\underbrace{(\langle E(\lambda) f, f \rangle)}_{\text{lebesgue measure associated to } m^f}$$

$$= \sum_{n:\, n\in B} \varphi(n)\, |\langle \phi_n, f \rangle|^2$$

$$m^{f,g}(B) = \sum_{n:\, n\in B} \overline{\langle \phi_n, f \rangle} \langle \phi_n, g \rangle$$

$$\int_B \varphi(\lambda)\, d(\langle E(\lambda) f, g \rangle)$$

$$= \sum_{n:\, n\in B} \varphi(n)\, \overline{\langle \phi_n, f \rangle} \langle \phi_n, g \rangle$$

$\longrightarrow$ PURE POINT SPECTRAL MEASURE.

Ex. B' $\quad (E(\lambda)f)(x) = \chi_{(-\infty,\lambda]}(x)\, f(x)$

$$F^f(\lambda) = \int_{-\infty}^{\lambda} |f(x)|^2\, dx \quad \Leftarrow \quad F \text{ abs. continuous with derivative } |f(\lambda)|^2$$

Spectral measure

$$m^f(B) = \int_B |f(\lambda)|^2\, d\lambda \qquad\qquad m^{f,g}(B) = \int_B \overline{f(\lambda)}\, g(\lambda)\, d\lambda$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\int \varphi(\lambda)\, |f(\lambda)|^2\, d\lambda \qquad\qquad \int \varphi(\lambda)\, \overline{f(\lambda)}\, g(\lambda)\, d\lambda$$

$\rightarrow$ abs. cont. spectral measure

$\square$

SPECTRAL THEOREM

Let $A, D(A)$ be a self-adj. op. in $\mathcal{H}$ with spectrum $\sigma(A)$.
Then there exist a unique spectral family $\{E_A(\lambda)\}$ s.t.

$$D(A) = \left\{ u \in \mathcal{H} : \int_{\sigma(A)} \lambda^2 \, d(\langle E_A(\lambda) u, u \rangle) < \infty \right\}$$

and

$$\langle Au, v \rangle = \int_{\sigma(A)} \lambda \, d(\langle E_A(\lambda) u, v \rangle) \qquad \forall u \in D(A),\ v \in \mathcal{H}$$

Moreover

$$\| Au \|^2 = \int_{\sigma(A)} \lambda^2 \, d(\langle E_A(\lambda) u, u \rangle) \qquad \forall u \in D(A)$$

INTERPRETATION Any self-adjoint operator can be reduced to a multiplication operator! However IS NOT CONSTRUCTIVE!

Spectral theorem for $Q_0, D(Q_0)$ in $L^2(\mathbb{R})$

let us consider the spectral family $(E(\lambda)f)(x) = \chi_{(-\infty,\lambda]}(x) f(x)$

then

$$\int \lambda \, d(\langle E(\lambda)u, v\rangle) = \int_{\mathbb{R}} dx \, x \, \overline{u}(x) v(x) = \langle Q_0 u, v\rangle$$

$\mathbb{R} = \sigma(Q_0)$

$$\int \lambda^2 d(\langle E(\lambda)u, u\rangle) = \int_{\mathbb{R}} x^2 |u(x)|^2 dx < \infty$$

condition appearing in $D(Q_0)$

$\Rightarrow$ $E(\lambda)$ is the spectral family for $Q_0, D(Q_0)$

Exercise: Find the spectral family for $(P_0 f)(x) = -i f'(x)$

Hint: P_0 is a mult. op. by k in Fourier space.

Ex $\quad H_D = -\Delta$ on $L^2((0,\pi))$, $\{\phi_n\}$ ONB $\quad$ L!S

Check $D(H_D) = \{ f : \sum_n n^4 |\langle \phi_n, f\rangle|^2 < \infty \}$ $\quad (*)$

Proof $\quad \phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx \quad n = 1, 2, \dots$ BON

$\lambda_n = n^2$

Spectral family: $\quad (E_{H_D}(\lambda) f)(x) = \sum_{n:\, n^2 \leq \lambda} \langle \phi_n, f\rangle \phi_n(x)$

Spectral measures

$$m^f(B) = \sum_{n:\, n^2 \in B} |\langle \phi_n, f\rangle|^2 \quad \Big/ \quad m^{f,g}(B) = \sum_{n:\, n^2 \in B} \overline{\langle \phi_n, f\rangle} \langle \phi_n, g\rangle$$

$(*)$ defines H_D as s.a. operator using <u>spectral thm</u>

STRATEGY

Classical model $H(p,q)$ $\xrightarrow[\text{procedure}]{1^{st}\text{ quant.}}$ Quantum Hamiltonian $H(\hat{x},\hat{p})$ on $L^2(\mathbb{R})$

which domain?

Start from $D(H) = \mathcal{S}(\mathbb{R})$

$\downarrow$

Build the self-adjoint extension $(H, D(H))$

SPECTRUM DECOMPOSITION

<u>Recall</u> the Lebesgue decomposition theorem.

Any Borel measure m on $\mathbb{R}$ can be decomposed

$$m = m_{pp} + m_{ac} + m_{sc}$$

m_{pp}: pure point measure

m_{ac}: abs. continuous w.r.t. Lebesgue

m_{sc}: singular cont. w.r.t. Lebesgue

$A, D(A)$ s.a. on $\mathcal{H}$; $\forall f \in \mathcal{H}$ $\quad \underbrace{\langle E_A(\lambda) f, f \rangle}_{\text{distrib. function}} \longrightarrow m_A^f$

m_A^f: associated Stieltjes measure

Lebesgue dec. thm suggests the following decomp. of $\mathcal{H}$

$\mathcal{H}_p(A) = \{ f \in \mathcal{H} : m_A^f \text{ is pure point} \}$

$\mathcal{H}_{ac}(A) = \{ f \in \mathcal{H} : m_A^f \text{ is abs. cont. w.r.t. Lebesgue} \}$

$\mathcal{H}_{sc}(A) = \{ f \in \mathcal{H} : m_A^f \text{ is sing cont. " " } \}$

$\mathcal{H}_c(A) = \{ f \in \mathcal{H} : m_A^f \text{ is continuous} \}$

Prop $\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A) = \mathcal{H}_p(A) \oplus \mathcal{H}_c(A)$

Moreover

$A|_{\mathcal{H}_p}$, $A|_{\mathcal{H}_{ac}}$, $A|_{\mathcal{H}_{sc}}$ are s.a. operators

on $\mathcal{H}_p$, $\mathcal{H}_{ac}$, $\mathcal{H}_{sc}$ respectively.

INTERPRETATION: any state is the sum of bound states, scattering states, and states with no phys. int.

SPECTRUM DECOMPOSITION

$$\sigma(A) = \sigma(A|_{\mathcal{H}_p(A)}) \cup \underbrace{\sigma(A|_{\mathcal{H}_{ac}(A)})}_{= \sigma_{ac}(A)} \cup \underbrace{\sigma(A|_{\mathcal{H}_{sc}(A)})}_{= \sigma_{sc}(A)}$$

<u>Rk</u> $\sigma(A|_{\mathcal{H}_p(A)}), \sigma_{ac}(A), \sigma_{sc}(A)$ are spectra of self adjoint operators
$\rightarrow$ closed subspaces of $\mathbb{R}$

<u>Recall</u> An accumulation point $\sigma_p(A)$ is not in general an eigenvalue, so $\sigma_p(A)$ might not be closed.

$$\sigma(A|_{\mathcal{H}_p(A)}) = \overline{\sigma_p(A)}$$

$\mathcal{H}_p(A)$ is the closed subspace spanned by all eigenvectors of A

Overall we have

$$\sigma(A) = \overline{\sigma_p(A)} \cup \sigma_c(A) = \overline{\sigma_p(A)} \cup \sigma_{ac}(A) \cup \sigma_{sc}(A)$$

Rk $\sigma_d(A)$ = discrete spectrum
= set of all eigenvalues with finite multiplicity & isolated points of $\sigma(A)$

$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A)$ essential spectrum

= $\sigma_c(A)$ + eigenvalues embedded in the cont. spectr. + eigenv. with ∞ mult.
+ accumulation points of eigenvalues.

FUNCTIONAL calculus

Prop. let $A, D(A)$ self adj. op., $\Phi : \mathbb{R} \to \mathbb{C}$ be continuous and bounded. Then $\exists$ a bounded operator $\Phi(A)$ s.t. $\forall u, v \in \mathcal{H}$

$$\boxed{\langle \Phi(A)u, v \rangle = \int_{\sigma(A)} \overline{\Phi(\lambda)}\, d(\langle E_A(\lambda)u, v \rangle)} \qquad (*)$$

Proof $\forall u \in \mathcal{H}$, we define the linear functional

$$\mathcal{F}_u : v \in \mathcal{H} \longrightarrow \int \overline{\Phi(\lambda)}\, d(\langle E_A(\lambda)u, v \rangle)$$

Check: $\mathcal{F}_u$ is bounded, by Riesz thm $\exists!\, h \in \mathcal{H} : \mathcal{F}_u(v) = \langle h, v \rangle$ and the map $\Phi(A) : u \longrightarrow h$ is a linear bounded op. in $\mathcal{H}$ satisfying $(*)$

Properties

- $\|\Phi(A)\| = \sup_{\lambda \in \sigma(A)} |\Phi(\lambda)|$

- $\Phi(A)^* = \overline{\Phi(A)}$

- $\phi(\lambda) = \alpha\,\phi_1(\lambda) + \beta\,\phi_2(\lambda) \qquad \forall\, \alpha, \beta \in \mathbb{C}$

  $\Rightarrow \Phi(A) = \alpha\,\Phi_1(A) + \beta\,\Phi_2(A)$

- $\phi(\lambda) = \phi_1(\lambda)\,\phi_2(\lambda)$

  $\Rightarrow \Phi(A) = \Phi_1(A)\,\Phi_2(A)$

Examples

1) $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$ RESOLVENT of A

$$\phi_z(\lambda) = \frac{1}{\lambda - z}$$ bounded and cont. f.

$$\phi_z(A) : \quad \langle \phi_z(A) u, v \rangle = \int_{\sigma(A)} \frac{d(\langle E_A(\lambda) u, v \rangle)}{(\lambda - z)} d\lambda$$

$$= (A - z)^{-1}$$

2) $\phi(\lambda) = e^{it\lambda}$

$\longrightarrow \quad e^{itA}$ exponential of the self adj. op. A

UNITARY GROUPS

<u>Def.</u> the family of linear operators $U(t)$, $t \in \mathbb{R}$ is a strongly continuous one-parameter group of unitary oper. in $\mathcal{H}$ if

i) $U(t)$ is a unitary op. in $\mathcal{H}$ $\forall\, t \in \mathbb{R}$ — one-parameter

ii) $U(t+s) = U(t)\,U(s)$ $\forall\, s,t \in \mathbb{R}$ — group property

iii) $\lim_{t \to 0} \| U(t)f - f \| = 0$ $\forall\, f \in \mathcal{H}$

continuity in $t=0$

$\Rightarrow$ continuity $\forall\, t \in \mathbb{R}$

PROPERTIES

$$\mathcal{U}(0) = \mathbb{1} \qquad \mathcal{U}(t)^{-1} = \mathcal{U}(-t)$$

Examples

$$(\mathcal{U}_a f)(x) = e^{iax} f(x) \quad a \in \mathbb{R} \quad \text{phase op.}$$

$$(V_b f)(x) = f(x+b) \quad b \in \mathbb{R} \quad \text{transl. op.}$$

References for this class: Sec. 4.8 Teta's book