

Lecture 2

The Formulation
of the theory

Classical Mech.

Quantum Mech.

L

1) STATE

$$(x, p) \in \mathbb{R}^{2d}$$

$$\psi(x) \in L^2(\mathbb{R}^d), \|\psi\|_2 = 1$$

" $\psi_t(x)$ "

2) EVOLUTION

$$t \rightarrow (x_t, p_t)$$

sol. of Ham. eq.

$t \rightarrow \psi(t, x)$ sol. of
Schröd. eq.

3) OBSERVABLES

$$f: (x, p) \rightarrow \mathbb{R}$$

possible outcomes

range of f

self-adjoint oper. on \mathcal{H}

$$A$$

4) PREDICTIONS

$f(x_t, p_t)$ with prob. 1

(DETERMINISTIC)

$$P(a \in I; \psi_t)$$

$$= \langle \psi_t, E_A(I) \psi_t \rangle$$

(PROBABILISTIC)

POSITION

$|\psi(x,t)|^2 = \text{prob. density}$

L^2

Hence

$$\langle x_k \rangle(t) = \int_{\mathbb{R}^d} x_k |\psi(x,t)|^2 dx \quad k=1, \dots, d$$

$$= \langle \psi_t, \hat{x}_k \psi_t \rangle$$

$\langle \cdot, \cdot \rangle$ scal. prod. *
(in $L^2(\mathbb{R}^d)$)

(also (\cdot, \cdot))

Position operator

$$\hat{x}_k : \psi(x) \longrightarrow x_k \psi(x)$$

$$\langle \psi_t | \hat{x}_k \psi_t \rangle$$

see next
page

* sc. p. orthogonality in the 1st arg. $(af, g) = \bar{a} (f, g)$

$$| \cdot \rangle = \text{ket} \quad , \quad \langle \cdot | = \text{bra}$$

Physics Book
Notations

scal prod. $\langle \cdot | \cdot \rangle$

Project onto $|\varphi\rangle$

$$P = |\varphi\rangle\langle\varphi|$$

$$P|\varphi\rangle = |\varphi\rangle \underbrace{\langle\varphi|\varphi\rangle}$$

$$\boxed{P\varphi = \langle\varphi, \varphi\rangle \varphi} \quad \leftarrow \text{we use}$$

or equivalently $(\varphi, \varphi) \varphi$

MULTIPLICATION OPERATOR

Take $\varphi \in L^\infty(\mathbb{R}^d)$

$$\langle \varphi \rangle(t) = \int \varphi(x) |\varphi(x,t)|^2 dx$$

$$= \langle \varphi_t, \hat{M}_\varphi \varphi_t \rangle$$

$$\hat{M}_\varphi: \varphi(x) \rightarrow \varphi(x)\varphi(x)$$

MOMENTUM OPER.

A nat. def. $\langle P_k \rangle(t) = \frac{d}{dt} m \langle x_k \rangle(t)$

by def. of $\langle x_k \rangle(t)$

$$= m \int dx x_k \frac{\partial}{\partial t} |\psi_t(x)|^2$$

int. by parts

Local conservation law

$$= -\frac{i\hbar}{2} \int dx x_k \operatorname{div} \mathbf{J} = m \int dx \bar{J}_k$$

$$\frac{\partial}{\partial t} |\psi_t(x)|^2 + \operatorname{div} \mathbf{J} = 0$$

$$= i\hbar \int dx \left(\psi_t \frac{\partial}{\partial x_k} \bar{\psi}_t - \bar{\psi}_t \frac{\partial}{\partial x_k} \psi_t \right)$$

$$\mathbf{J} = \frac{i\hbar}{2m} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi)$$

for ψ fast decay

$$= -i\hbar \int dx \bar{\psi}_t(x) \frac{\partial}{\partial x_k} \psi_t(x)$$

$$= \langle \psi_t, \frac{\hbar}{i} \frac{\partial}{\partial x_k} \psi_t \rangle$$

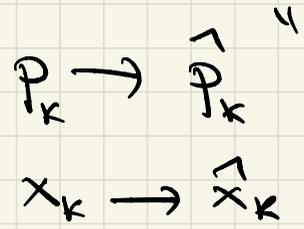
$$\hat{x}_k : \psi(x) \rightarrow x_k \psi(x) \quad (1)$$

$$\hat{p}_k : \psi(x) \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial x_k} \psi(x) \quad (2)$$

1st quantization procedure

ENERGY OP.

$$H(x, p) = \frac{p^2}{2m} + V(x)$$



(1) & (2)

$$\hat{H} = -\frac{\hbar^2}{2m} \Delta + V(x)$$

$$\langle E \rangle(t) = \langle \psi_t, \hat{H} \psi_t \rangle$$

Time dep. Schröd eq.

$$i \frac{\partial}{\partial t} \psi(x, t) = \hat{H} \psi(x, t)$$

Few examples

$$\hat{x}_k, \hat{p}_k, \hat{H}, \hat{M}_\varphi$$

L7

LINEAR & symmetric

NICE! eigenvalues of symm.
operators are real

For $\mathcal{X} = L^2(\mathbb{R}^d)$, $\varphi \in L^\infty(\mathbb{R}^d)$, $M_\varphi : (M_\varphi f)(x) = \varphi(x) f(x)$ ^{L8}

M_φ IS BOUNDED

$$\|M_\varphi f\| \leq \|\varphi\|_{L^\infty} \|f\|$$

$$\|M_\varphi\| = \sup_{\substack{f \in L^2(\mathbb{R}^d) \\ f \neq 0}} \frac{\|M_\varphi f\|}{\|f\|} = \|\varphi\|_{L^\infty} < \infty$$

A bounded op. $(A, D(A))$ $D(A) \subset \mathcal{X}$ can be extended to the whole \mathcal{X} in a unique way \Rightarrow
 $D(A) = \mathcal{X}$

Claim $\mathcal{H} = L^2(\mathbb{R})$, $(Q_0 f)(x) = x f(x)$, $D(Q_0) = \mathcal{D}(\mathbb{R})$ ^{L9}

$\Rightarrow (Q_0, D(Q_0))$ is unbounded

Proof $\exists f_n \in \mathcal{D}(\mathbb{R}) : \|f_n\|_2 = 1$ s.t. $\|Q_0 f_n\| \xrightarrow{n \rightarrow +\infty} \infty$

Pick $f_n(x) = \frac{1}{\sqrt{\pi}} e^{-\frac{(x-n)^2}{2}}$, $\|f_n\|_2^2 = \frac{1}{\sqrt{\pi}} \int dx e^{-\frac{(x-n)^2}{2}} = 1$

$$\|Q_0 f_n\|^2 = \frac{1}{\sqrt{\pi}} \int dx x^2 e^{-\frac{(x-n)^2}{2}} = \frac{1}{\sqrt{\pi}} \int_{z=x-n} dz (z+n)^2 e^{-z^2}$$

$\xrightarrow{n \rightarrow +\infty} +\infty$

MO.M. OPERATOR

$$(P_0 f)(x) = -i f'(x) \quad \mathcal{H} = L^2(\mathbb{R})$$

$(P_0, D(P_0) = \mathcal{H}(\mathbb{R}))$ is unbounded

Proof

$$f_n(x) = (\mathcal{F}^{-1} f_n^2)(x)$$

$$f_n^2(k) = \frac{1}{\pi^{1/4}} e^{-\frac{(k-n)^2}{2}}$$

For unbounded op.

$(A, D(A))$

Observ. in QM
will be described
by self-adjoint
operators

$$A = A^* \quad \& \quad D(A) = D(A^*)$$

POSITION OP.

$$(Q_k \psi)(x) = x_k \psi(x)$$

$$D(Q_k) = \left\{ \psi \in L^2(\mathbb{R}^d) \mid \int dx |x_k \psi(x)|^2 < \infty \right\}$$

if A s.o.

$$\sigma(A) \subseteq \mathbb{R}$$

Proof: TUTORIAL
class

Monday 15-17

MOM. OPERATOR

$$(P_k \psi)(x) = \frac{\hbar}{i} \frac{\partial \psi}{\partial x_k}(x)$$

$$D(P_k) = \left\{ \psi \in L^2(\mathbb{R}^d) \mid \int dp |P_k \tilde{\psi}(p)|^2 < \infty \right\}$$

COMMUTATION RELATIONS

L12

$$[\hat{x}_k, \hat{p}_e] \psi = \hat{x}_k \hat{p}_e \psi - \hat{p}_e \hat{x}_k \psi$$

$$= x_k \frac{\hbar}{i} \frac{\partial}{\partial x_e} \psi - \frac{\hbar}{i} \frac{\partial}{\partial x_e} (x_k \psi) = -\frac{\hbar}{i} \delta_{k,e} \psi$$

$$[\hat{x}_k, \hat{p}_e] \psi = i\hbar \delta_{k,e} \mathbb{1} \psi$$

Physical meaning?

$$\Delta x_k(t) \Delta p_e(t) \geq \frac{\hbar}{2} \delta_{k,e}$$

HEISENBERG

PRINCIPLE (1927)

→ It is essentially
impossible to follow trajectories!

Prop. Let A, B, C be linear, symm. op. on \mathcal{X} , let $\psi \in \mathcal{X}$ s.t. $\| \psi \| = 1$

$$[A, B] \psi = i C \psi$$

Denote $\Delta A^2 = \langle \psi, (A - \langle A \rangle)^2 \psi \rangle$ and ΔB^2 similarly, then

$$\Delta A \cdot \Delta B \geq \frac{1}{2} | \langle C \rangle |$$

Proof. $\Delta A \cdot \Delta B = \| (A - \langle A \rangle) \psi \| \| (B - \langle B \rangle) \psi \| \stackrel{CS}{\geq} | \langle (A - \langle A \rangle) \psi, (B - \langle B \rangle) \psi \rangle |$

symm. \curvearrowright

$$= | \langle \psi, \underbrace{(A - \langle A \rangle)(B - \langle B \rangle)}_{\substack{= F + \frac{i}{2} C \\ F, C \text{ symmetric}}} \psi \rangle |$$

$$\frac{1}{2} [(A - \langle A \rangle)(B - \langle B \rangle) + (B - \langle B \rangle)(A - \langle A \rangle)] + \frac{1}{2} [A, B]$$

hence $\Delta A \cdot \Delta B \geq | \langle F \rangle + \frac{i}{2} \langle C \rangle |$

mean value of sym. operators is real

$$= \left(\langle F \rangle^2 + \frac{1}{4} \langle C \rangle^2 \right)^{1/2} \geq \frac{1}{2} | \langle C \rangle | \quad \blacksquare$$

the minimum uncertainty is reached when

$$\begin{cases} (B - \langle B \rangle) \psi = \lambda (A - \langle A \rangle) \psi & \lambda \in \mathbb{R} \\ \langle (A - \langle A \rangle)^2 \rangle = 0 \end{cases} \Leftrightarrow \langle F \rangle = 0$$

Ex

$d=1$

$$\Delta x \Delta p = \hbar/2$$

$\beta > 0$

$$\psi_{\beta}(x) = \left(\frac{\beta}{\pi \hbar} \right)^{1/4} e^{-\frac{\beta}{2\hbar} (x-x_0)^2 + \frac{i}{\hbar} p_0 x}$$

coherent state

$\langle x \rangle = x_0$

$\langle p \rangle = p_0$

Ex. Show that

$$(\Delta x)^2 = \frac{\hbar}{2\beta}, \quad (\Delta p)^2 = \frac{\hbar\beta}{2}$$

$$\beta = \frac{\Delta p}{\Delta x}$$

PREDICTIONS let ψ_t the state of the system at time t , LIS

$$\psi_t \in L^2(\mathbb{R}),$$

$$(Q, D(Q)) \quad \mathbb{P}(x \in I; \psi_t) = \int_I dx |\psi_t(x)|^2$$

prob. of find.
a particle in
a point = 0

$$(P, D(P)) \quad \mathbb{P}(p \in I; \psi_t) = \int_I \frac{dp}{h} |\tilde{\psi}_t(p/h)|^2$$

$$(A, D(A)) \quad \mathcal{U}(A) = \{a_i, \text{non degenerate}\}$$

$$A\varphi_i = a_i\varphi_i, \quad \{\varphi_i\} \text{ basis in } \mathcal{X}$$

$$\rightarrow \mathbb{P}(A \in I; \psi_t) = \sum_{n: a_n \in I} |\langle \varphi_n, \psi_t \rangle|^2$$

prob. to
find as
an outcome
of the
measure
the eigenv. corr. to φ_n

Note $\mathbb{P}(A = a_i; \varphi_i) = 1$

For a generic observable Q described by \hat{Q} self-adj. operator \underline{A} we have 216

$$P(Q \in I \subseteq \mathbb{R}; \psi_t) = \langle \psi_t, E_A(I) \psi_t \rangle$$

where $\{E_A(\lambda)\}_{\lambda \in \mathbb{R}}$ the spectral family associated to the operator A

$f \in \mathcal{X}, \lambda \in \mathbb{R} \rightarrow$ next Monday 9-11

$$\leadsto F^f(\lambda) = \langle E(\lambda) f, f \rangle$$

Distribution function

1) Physical consequences of the linearity of the space 117

$$\psi_1, \psi_2 \in \mathcal{H}$$

$$\|\psi_1\|_2, \|\psi_2\|_2 = 1$$

$$\Rightarrow \psi(x) = \frac{\psi_1(x) + \psi_2(x)}{\|\psi_1 + \psi_2\|_2}$$

is also a possible state

$$|\psi(x)|^2 = c \left(|\psi_1(x)|^2 + |\psi_2(x)|^2 + 2 \operatorname{Re} \overline{\psi_1(x)} \psi_2(x) \right)$$

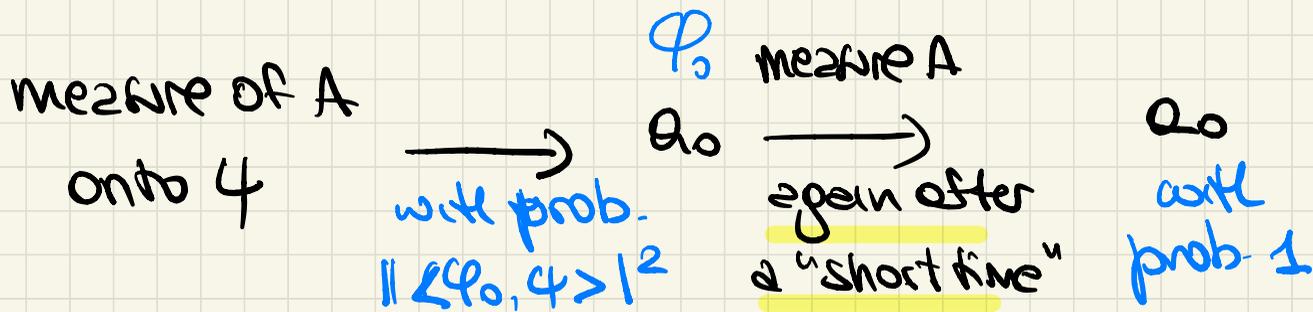
↙ purely quantum effect

→ INTERFERENCE TERM ←

the prob. distribution of $\psi_1 + \psi_2$ is NOT the sum of the prob. distributions of having the particle either in ψ_1 or in ψ_2

2) EFFECT OF MEASURE (in the standard Copenhagen interpretation)^{L18}

Guiding idea \mathcal{Q} repr. by A s.t. $\sigma(A) = \{a_i\}$
and ψ not eigenfunction of A



INTERPRETATION

transition from ψ to ψ_0 (COLLAPSE OF THE WAVE FUNCTION)

the 1st measure has produced an instantaneous

STOCHASTIC + NON LINEAR

Measure -

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INTERACTION

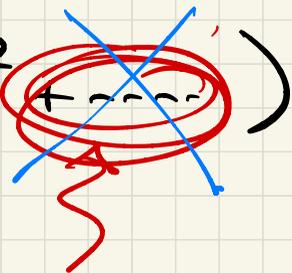
QUANTUM + CLASSICAL object

SCHRÖDINGER CAT PROBLEM: superposition states are transferred to the measurement device

$$I = I_1 + I_2$$

$$I^2 = (I_1^2 + I_2^2 + \dots)$$

microscopic shot



For all Practical Purposes (FAPP) Bell '63
the collapse of wave function works

References

Contents of this class (in a more extended form) can be found in

Sections 3.4 & 3.5 }
Chapter 5 } Alessandro Teta's
book

BACKUP

Def. The operator B defined on $D(B)$ is an extension of A defined on $D(A)$ if

$$D(B) \supseteq D(A)$$

and

$$Bf = Af \quad \forall f \in D(A)$$

Notation: $B \supseteq A$ or $B \supset A$ if $D(B) \neq D(A)$