# Homework 4

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### Policies

These homeworks are not mandatory for this course but you are highly encouraged to try to solve them and any work in this direction will be highly appreciated. You may discuss your homework problems freely with other students, but please refrain from looking at their code or writeups. Please submit your solutions using latex to write your solutions in pdf format.

### Questions

(1) Let $G$ be a matrix of the form

$$G = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & * & \ddots & & \\ & & \vdots & & \ddots & \\ & & * & & & 1 \end{bmatrix}$$

where $*$ denote a nonzero. Show that $G^{-1}$ has the same structure as $G$.

(2) In the previous question, find a precise formula for the inverse $G^{-1}$ given that the nonzero elements on the $k$ column of $G$ are $[1, a_1, \dots, a_{n-k}]$.

(3) For a symmetric positive definite matrix $A$ consider the following decomposition

$$A = A_1 = \begin{bmatrix} a_{11} & u^T \\ u & \tilde{A} \end{bmatrix} = \begin{bmatrix} \sqrt{a_{11}} & \\ \frac{1}{\sqrt{a_{11}}} u & I \end{bmatrix} \begin{bmatrix} 1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & \frac{1}{\sqrt{a_{11}}} u^T \\ & I \end{bmatrix}$$

where $A_2 = \tilde{A} - \frac{1}{a_{11}} u u^T$ and $\tilde{A}$ is the $(n-1) \times (n-1)$ bottom-right submatrix of $A$. Applying this decomposition recursively on the sequence $A_1, A_2, \dots$ prove the following statement:

**Thm.** Let $A = LL^T$ be the Cholesky decomposition of the symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$. Then $L_{ij} = 0$ for all $i, j \in \{1, \dots, n\}$ such that $i > j$ or $i, j \notin \mathrm{env}(A)$.

(4) Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & \\ \vdots & & \ddots & \\ 1 & & & 1 \end{bmatrix}$$

and find $\alpha$ such that $A + \alpha I$ is positive definite.

(5) (∗) For a diagonalizable matrix $A$ and for a function $f : \mathbb{C} \longrightarrow \mathbb{C}$, define

$$f(A) := X \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} X^{-1}$$

Show that for any diagonalizable matrix $A$ and any function $f : \mathbb{C} \longrightarrow \mathbb{C}$ there exists a polynomial $p : \mathbb{C} \longrightarrow \mathbb{C}$ of degree at most $n$ such that $f(A) = p(A)$.