Homework 2

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Policies

These homeworks are not mandatory for this course but you are highly encouraged to try to solve them and any work in this direction will be highly appreciated. You may discuss your homework problems freely with other students, but please refrain from looking at their code or writeups. Please submit your solutions using latex to write your solutions in pdf format.

Questions

(1) A matrix norm is induced by a vector norm if it is defined as

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Prove that for all induced matrix norms it holds $\rho(A) \leq \|A\|$. You can also just consider the case of Hermitian matrices.

(2) Let K, H be two subspaces and let P_K, P_H be the corresponding orthogonal projections. Define

$$\sin\theta(K,H) = \min_{y \in K} \max_{x \in H} \frac{\|y - x\|_2}{\|y\|_2}\,.$$

Show that $\sin\theta(K,H) = \|P_{K^\perp} P_H\|_2 = \|P_K - P_H\|_2$.

(3) Show that for any triangular matrix T there exists a triangular matrix T_ε that has all distinct eigenvalues and is such that $\|T - T_\varepsilon\|_2 \leq \varepsilon$. Thus, use the Schur decomposition theorem to prove that the set of diagonalizable matrices is dense in $\mathbb{C}^{n\times n}$.

(4) Prove that $\mathrm{Range}(A) = \mathrm{Null}(A^*)^\perp$