Homework 4

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Policies

These homeworks are not mandatory for this course but you are highly encouraged to try to solve them and any work in this direction will be highly appreciated. You may discuss your homework problems freely with other students, but please refrain from looking at their code or writeups. Please submit your solutions using latex to write your solutions in pdf format.

Questions

1. Consider the ODE $y' = f(x)$ with $f(x) = (k+1)x^k$. This differential equation is integrated exactly by the $(k+1)$-step explicit Adams method. Thus

$$y(x_k) - y(x_{k-1}) = h \sum_{j=0}^{k} \gamma_j \nabla^j f_{k-1}\,.$$

Using this formula (and observing that $\gamma_k \neq 0$) show that the order of the $\boldsymbol{k}$**-step** explicit Adams method is not greater than k.

2. Use the identities

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad -\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^n}{n}$$

to show that the coefficients

$$\gamma_j^* = (-1)^j \int_0^1 \binom{-\lambda+1}{j} d\lambda$$

of the implicit Adams method satisfy the recursion

$$0 = \gamma_m^* + \frac{1}{2}\gamma_{m-1}^* + \frac{1}{3}\gamma_{m-2}^* + \cdots + \frac{1}{m+1}\gamma_0^*$$

3. Show that all the roots of the first characteristic polynomial of a $LMM(k)$method with $\alpha\_j = -\alpha\_k - j$, for all j, are simple and lie on the unit circle.

4. Consider the companion matrix of the first characteristic polynomial of $LMM(k)$

$$\varrho(\xi) = \alpha_k \xi^k + \cdots + \alpha_1 \xi + \alpha_0 \longleftrightarrow C_\varrho = \begin{bmatrix} -\frac{\alpha_{k-1}}{\alpha_k} & -\frac{\alpha_{k-2}}{\alpha_k} & \cdots & -\frac{\alpha_0}{\alpha_k} \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}$$

If the $LMM(k)$ is zero-stable then the roots of ϱ are either on the unit circle and have multiplicity one or they are inside the unit circle. Thus we can write

$$C_\varrho = XJX^{-1} \qquad \text{with} \qquad J = \mathrm{Diag}\Big(\dots, \lambda_i, \dots, \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_j \end{bmatrix}, \dots\Big)$$

where the diagonal entries correspond to roots lying on the unit circle while the diagonal blocks correspond to those inside the circle.

Show that there exists a matrix norm $\|\cdot\|$ such that $\|C_\varrho\| \le 1$.

Hint. It can help recalling that (a) $\rho(M)$ is the inf of all the norms $\|M\|$, for any matrix M; and that (b) $\|y\|_M = \|My\|$ is a vector norm for $M \in \{X, X^{-1}\}$ and any vector norm $\|\cdot\|$.