Homework 3

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Policies

These homeworks are not mandatory for this course but you are highly encouraged to try to solve them and any work in this direction will be highly appreciated. You may discuss your homework problems freely with other students, but please refrain from looking at their code or writeups. Please submit your solutions using latex to write your solutions in pdf format.

Questions

(1) A simple graph (at most one edge between any two pair of nodes) $G = (V, E)$ is strongly connected if for any two nodes $i, j \in V$ there exists a path of edges in E that goes from i to j. Let $A \geq 0$ be a $n \times n$ matrix with nonnegative entries. Show that A is irreducible if and only the graph $G = (V, E)$ defined by $V = \{1, \dots, n\}$ and $E = \{ij : A_{ij} \neq 0\} \subseteq V \times V$ is strongly connected.

(2) Show that for an invertible matrix U we have $\mathrm{rank}(AU) = \mathrm{rank}(A) = \mathrm{rank}(UA)$

(3) Let $A \in \mathbb{C}^{n \times n}$ and $x \in \mathbb{C}^n$. Show that
- $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$
- $\|A\|_2 \leq \sqrt{n}\|A\|_\infty$
- For any unitary matrix U it holds $\|Ux\|_2 = \|x\|_2$
- For any unitary matrix U it holds $\|UA\|_2 = \|A\|_2$
- $\|A\|_2 = \sqrt{\rho(A^*A)}$ where $\rho(M) = \max_{\lambda \in \sigma(M)} |\lambda|$ is the spectral radius of M

(4) Show that if $Q \in \mathbb{C}^{n \times p}$ has linear independent columns, then Q^*Q is positive definite and QQ^* is positive semi-definite.

(5) Let A be Hermitian and let $r(x) = x^*Ax/(x^*x)$. Suppose $V \subseteq \mathbb{C}^n$ is a subspace such that $x \in V$ implies $\nabla r(x) \in V$. Show that V is invariant for A.