GSSI-SISSA joint lectures "Mathematical methods in Field theory and quantum mechanics"

Lecture 1 – 2020.06.29 – 15:00 via Zoom

M. Gubinelli and M. Hofmanova. "A PDE Construction of the Euclidean Φ^4_3 Quantum Field Theory." *ArXiv:1810.01700*, October 3, 2018. http://arxiv.org/abs/1810.01700.

Topic of the lectures: what does it means "stochastic quantisation".

It is a too to study measures on spaces of functions (or better distributions) on $\mathbb{R}^d$ which are "complicated".

(I will not talk about KPZ equation).

1 The Φ^4_d model

This should be the probability measure ν on $\mathscr{S}'(\mathbb{R}^d)$ $\varphi : \mathbb{R}^d \to \mathbb{R}$

$$\nu(\mathrm{d}\varphi) \propto \exp\left(-2\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla\varphi|^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4\right)\mathrm{d}x\right)\mathrm{D}\varphi$$

This expression is wrong in many ways. The important features that "motivate" this naive expression is that you want that this measure has the properties

- Translation/rotation invariant
- "non-trivial" (for us this just means: non-Gaussian)
- Is reflection positive. i.e. certain averages of a specific form has to be positive

There three properties make this kind of measures relevant in applications to QFT.

More rigorously one can introduce a lattice structure on $\mathbb{R}^d$, i.e. $(\varepsilon\mathbb{Z})^d$ and moreover limit the domain to a finite region. I.e. we replace $\mathbb{R}^d$ with a finite lattice $\Lambda_{\varepsilon,M} = ([-M,M]\cap(\varepsilon\mathbb{Z}))^d$ (with periodic boundary conditions). We consider the measure on configurations $\varphi : \Lambda_{\varepsilon,M} \to \mathbb{R}$

$$\nu_{\varepsilon,M}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,M}}\exp\left(-2\varepsilon^d\sum_{x\in\Lambda_{\varepsilon,M}}\left(\frac{1}{2}|\nabla_\varepsilon\varphi_x|^2 + \frac{m^2}{2}\varphi_x^2 + \frac{\lambda}{4}\varphi_x^4\right)\right)\left(\prod_{x\in\Lambda_{\varepsilon,M}}\mathrm{d}\varphi_x\right)$$

Our goal is to find mathematical tools to control the family of measures $(\nu_{\varepsilon,M})$ as $\varepsilon \to 0$ (UV limit) and $M \to \infty$ (infinite volume limit).

We concentrate in $d = 3$ ($d = 1, 2$ are easier and I cannot do $d = 4$).

If $\lambda = 0$ we have a family of Gaussian measures for which taking the two limits is not difficult because everything reduces to control the covariance. In that case we call the limit measure μ and is the centred Gaussian measure on $\mathscr{S}'(\mathbb{R}^3)$ (Schwartz distributions) with covariance

$$\int_{\mathscr{S}'(\mathbb{R}^3)}\varphi(f)\varphi(g)\mu(\mathrm{d}\varphi) = \langle f, (m^2 - \Delta)^{-1} g\rangle_{L^2(\mathbb{R}^3)} = \int \mathrm{d}k\overline{\hat{f}(k)}(m^2 + k^2)^{-1}\hat{g}(k).$$

Or equivalently

$$\varphi(f) \sim_\mu \mathcal{N}(0, \langle f, (m^2 - \Delta)^{-1} f\rangle_{L^2(\mathbb{R}^3)}).$$

There is a machine (Osterwalder–Schader recontruction) to convert any probability measure which satisfies

Euclidean invariance + Reflection positivity + some regularity

into a $d-1$ dimensional QFT: i.e. Hilbert space, operators which commute under appropriate conditions, representation of the Poincaré group,... ($\Delta_{\mathbb{R}^d} \to \Box_{\mathbb{R}^{d-1}}$, $x_1 \to it$).

If you apply the machine to μ then you get a QFT which represent non-interacting particles. I.e. not those you see in the experiments.

Life would be much easier if we could consider the measures

$$\frac{1}{Z_{\varepsilon,M}} \exp\left(-2\varepsilon^d \sum_{x \in \Lambda_{\varepsilon,M}} \left(\frac{\alpha}{2}|\Delta_\varepsilon \varphi|^2 + \frac{1}{2}|\nabla_\varepsilon \varphi|^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4\right)\right)\left(\prod_{x \in \Lambda_{\varepsilon,M}} \mathrm{d}\varphi_x\right)$$

for some $\alpha > 0$ (since this measure has better UV properties, i.e. milder small scale oscillations). But this measure is not reflection positive.

Interesting variation is however the following class of measures

$$\frac{1}{Z_{\varepsilon,M}} \exp\left(-2\varepsilon^d \sum_{x \in \Lambda_{\varepsilon,M}} \left(\frac{1}{2}||i\nabla_\varepsilon|^\gamma \varphi|^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4\right)\right)\left(\prod_{x \in \Lambda_{\varepsilon,M}} \mathrm{d}\varphi_x\right)$$

with $\gamma \in (3/4, 1)$ where the operaror $|i\nabla_\varepsilon|^\gamma$ is a fractional (discrete derivative) defined via Fourier trasform. This class of models is reflection positive.

The key point is that each of the measures $\nu_{\varepsilon,M}$ (or variants) is reflection positive and translation invariant at least for the lattice translations. (this is one of the reasons to consider periodic bc). And these properties pass to the limit.

Rotation invariance has to be established on the limit since it is violated on the lattice.

2 Short distance properties

Now the goal is to understadn the limit of the family

$$\nu_{\varepsilon,M}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,M}} \exp\left(-2\varepsilon^d \sum_{x \in \Lambda_{\varepsilon,M}} \left(\frac{1}{2}|\nabla_\varepsilon \varphi|^2 + \frac{m^2 + a_\varepsilon}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4\right)\right)\left(\prod_{x \in \Lambda_{\varepsilon,M}} \mathrm{d}\varphi_x\right)$$

$$= \frac{1}{Z_{\varepsilon,M}} \exp\left(-2\varepsilon^d \sum_{x \in \Lambda_{\varepsilon,M}} \left(\frac{a_\varepsilon}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4\right)\right) \underbrace{\exp\left(-2\varepsilon^d \sum_{x \in \Lambda_{\varepsilon,M}} \left(\frac{1}{2}|\nabla_\varepsilon \varphi|^2 + \frac{m^2}{2}\varphi^2\right)\right)\left(\prod_{x \in \Lambda_{\varepsilon,M}} \mathrm{d}\varphi_x\right)}_{Z^0_{\varepsilon,M}\mu_{\varepsilon,M}(\mathrm{d}\varphi)}$$

possibly by tuning the parameter a_ε along the way as it will be needed.

The main problem: as $M \to \infty$ or as $\varepsilon \to 0$, $\nu_{\varepsilon,M}$ become singular wrt. $\mu_{\varepsilon,M}$. The Gaussian measure $\mu = \lim_{\varepsilon,M} \mu_{\varepsilon,M}$ does not give a reasonable approximation of a possible limit ν.

Underlying difficulty: what is an "effective" description of a possible limit ν. Some techniques: Fourier transform/moments/generating function (e.g. Gaussian), using a density wrt. another measure, using a Gibbsian specification, using integration by parts, using various kind of expansions: cluster expansion / phase cell expansion.

Example 1. If $d = 2$ then if one takes $a_\varepsilon \approx -(\log \varepsilon^{-1})^2 \to +\infty$ as $\varepsilon \to \infty$ then $\nu_M = \lim_{\varepsilon \to 0} \nu_{M,\varepsilon}$ is absolutely continuous wrt. the Gaussian measure $\mu_M = \lim_{\varepsilon \to 0} \mu_{M,\varepsilon}$ and one can make sense of the formula

$$\frac{\mathrm{d}\nu_M}{\mathrm{d}\mu_M} = \exp\left(-2\lambda \int_{\mathbb{T}_M^2} [\![\varphi^4(x)]\!] \mathrm{d}x \right)$$

where $[\![\varphi^4(x)]\!] := \varphi^4(x) - \infty \varphi^2(x) - \infty$ is the Wick product of $\varphi(x)^4$. The difficulty here is linked to the fact that under μ the r.v. $\varphi \in \mathscr{S}'(\mathbb{T}_M^2)$ is not a function and has negative regularity, indeed

$$\int_{\mathscr{S}'(\mathbb{T}^2)} \varphi(f)^2 \mu(\mathrm{d}\varphi) \approx \sum_{k \in \mathbb{Z}^2} |\hat{f}(k)|^2 (m^2 + k^2)^{-1}.$$

so in particular $\varphi(x) = \varphi(\delta_x)$ does not exist.

$$[\![\varphi^4(x)]\!] = \mu - \lim_{\varepsilon \to 0} (\varphi(\rho_\varepsilon(\cdot + x))^4 - 6\mathbb{E}(\varphi(\rho_\varepsilon(\cdot + x))^2)\varphi(\rho_\varepsilon(\cdot + x))^2 - \mathbb{E}(\varphi(\rho_\varepsilon(\cdot + x))^4)),$$

where $\rho_\varepsilon(x) = \varepsilon^{-2}\rho(x/\varepsilon)$ is a sequence of smooth functions such that $\rho_\varepsilon \to \delta$. Note that

$$\mathbb{E}(\varphi(\rho_\varepsilon(\cdot + x))^2) \approx \log(\varepsilon^{-1}).$$

With some more computation one can prove that $\varphi \in \mathscr{C}^{-\kappa} = B_{\infty,\infty}^{-\kappa}$ almost surely for any $\kappa > 0$, i.e. we say that $f \in \mathscr{C}^\alpha$ if

$$\|f\|_{\mathscr{C}^\alpha} = \sup_{i \geqslant 0} 2^{i\alpha} \|\Delta_i f\|_{L^\infty} < \infty$$

where $\Delta_i f$ filter the Fourier transform of f in such a way to leave only components with wavevector $k \sim 2^i$. In our case we have, under μ

$$\|\Delta_i \varphi\|_{L^\infty} \lesssim 2^{i\kappa}$$

for any small $\kappa > 0$. The Wick powers $x \mapsto [\![\varphi^n(x)]\!]$ are defined for any $n \geqslant 1$ as random variables in $\mathscr{C}^{-\kappa'}$ for some $\kappa' > 0$. The field φ under the measure μ is called the Gaussian free field.

For more informations, refere to the book of Simon $P(\varphi)_2$:

Simon, Barry. *P(ϕ)2 Euclidean (Quantum) Field Theory*. Princeton, N.J: Princeton University Press, 1974.

When $d=3$ we have that the Gaussian free field φ has regularity

$$\int_{\mathscr{S}'(\mathbb{T}^3)} |\Delta_i \varphi(x)|^2 \mu(\mathrm{d}\varphi) \approx \sum_{k\in\mathbb{Z}^3:2^i\lesssim|k|\lesssim 2^{i+1}} (m^2+k^2)^{-1} \approx 2^i$$

which can be shown to lead to $\varphi \in \mathscr{C}^{-1/2-\kappa}$ since $\Delta_i\varphi(x) \approx 2^{i(1/2+\kappa)}$ for some small κ (essentially due to a Borel-Cantelli argument). So $d=3$ is a more singular problem than $d=2$. In particular one has

$$[\![\varphi^2]\!] \in \mathscr{C}^{-1-\kappa},$$

and $[\![\varphi^3]\!]$ does not exists, meaning $\mathbb{E}(([\![\varphi^3]\!](f))^2) = +\infty$ for any nice test function $f \in \mathscr{S}(\mathbb{T}^3)$.

What will happen is that ν_M will be singular wrt. μ_M (in $d=3$).

We expect to need to renormalize to be able to take the limit and we also do not expect to be able to write down a density for ν_M wrt. any nice or simple mesure.

3 Stochastic quantisation

Now we look at a stochastic dynamics / stochastic equation which has $\nu_{\varepsilon,M}$ as invariant measure

$$\nu_{\varepsilon,M}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,M}} \exp(-2V_{\varepsilon,M}(\varphi)) \left(\prod_{x\in\Lambda_{\varepsilon,M}} \mathrm{d}\varphi_x\right)$$

$$V_{\varepsilon,M}(\varphi) = \varepsilon^d \sum_{x\in\Lambda_{\varepsilon,M}} \left(\frac{1}{2}|\nabla_\varepsilon\varphi|^2 + \frac{m^2+a_\varepsilon}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4\right)$$

One such equations is the following:

$$\mathrm{d}\psi(t) = -\frac{\partial V(\psi(t))}{\partial\varphi}\mathrm{d}t + \mathrm{d}B(t),$$

where $\psi: \mathbb{R}_+ \to \mathbb{R}^{\Lambda_{\varepsilon,M}}$ is a stochastic proceed and B is a Brownian motion on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$, i.e

$$\mathbb{E}[B_x(t)B_y(s)] = \varepsilon^{-d}(t\wedge s)\delta_{x,y}, \qquad t,s \geqslant 0, x,y \in \Lambda_{\varepsilon,M}.$$

$$\frac{\partial V(\varphi)}{\partial \varphi_x} = (\Delta_{\Lambda_{\varepsilon,M}}\varphi)_x + (m^2+a_\varepsilon)\varphi_x + \lambda\varphi_x^3.$$

It is a fact that this equation has global solution and that $\nu_{\varepsilon,M}$ is the only invarian measure, meaning that if Law $(\psi(0)) = \nu_{\varepsilon,M}$ then Law $(\psi(t)) = \nu_{\varepsilon,M}$ for any $t \geqslant 0$ and moreover for any reasonable intial condition we have Law $(\psi(t)) \to \nu_{\varepsilon,M}$.

$$\nu_{\varepsilon,M} \sim \psi(t) = F(B).$$