GSSI-SISSA joint lectures "Mathematical methods in Field theory and quantum mechanics"

Lecture 2 – 2020.07.03 – 15:00 via Zoom

M. Gubinelli and M. Hofmanova. "A PDE Construction of the Euclidean Φ_3^4 Quantum Field Theory." *ArXiv:1810.01700*, October 3, 2018. http://arxiv.org/abs/1810.01700.

1 Stochastic quantisation

Now we look at a stochastic dynamics / stochastic equation which has $\nu_{\varepsilon,M}$ as invariant measure

$$\nu_{\varepsilon,M}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,M}} \exp(-2V_{\varepsilon,M}(\varphi)) \left(\prod_{x \in \Lambda_{\varepsilon,M}} \mathrm{d}\varphi_x \right)$$

$$V_{\varepsilon,M}(\varphi) = \varepsilon^d \sum_{x \in \Lambda_{\varepsilon,M}} \left(\frac{1}{2} |\nabla_\varepsilon \varphi|^2 + \frac{m^2 + a_\varepsilon}{2} \varphi^2 + \frac{\lambda}{4} \varphi^4 \right)$$

recall $\Lambda_{\varepsilon,M} = (\varepsilon \mathbb{Z} \cap [-M, M])^d$ with $\varepsilon = 2^{-N_1}$ and $M = 2^{N_2}$. With $\varepsilon \to 0$, $M \to \infty$.

One such equations is the following SDE:

$$\mathrm{d}\psi(t) = -\frac{\partial V(\psi(t))}{\partial \varphi} \mathrm{d}t + \mathrm{d}B(t),$$

where $\psi: \mathbb{R}_+ \to \mathbb{R}^{\Lambda_{\varepsilon,M}}$ is a stochastic proceed and B is a Brownian motion on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$, i.e

$$\mathbb{E}[B_x(t) B_y(s)] = \varepsilon^{-d} (t \wedge s) \delta_{x,y}, \qquad t, s \geqslant 0, x, y \in \Lambda_{\varepsilon,M}.$$

$$\frac{\partial V(\varphi)}{\partial \varphi_x} = -(\Delta_{\Lambda_{\varepsilon,M}} \varphi)_x + (m^2 + a_\varepsilon) \varphi_x + \lambda \varphi_x^3.$$

It is a fact that this equation has global solution and that $\nu_{\varepsilon,M}$ is the only invariant measure, meaning that if Law $(\psi(0)) = \nu_{\varepsilon,M}$ then Law $(\psi(t)) = \nu_{\varepsilon,M}$ for any $t \geqslant 0$ and moreover for any reasonable intial condition we have Law $(\psi(t)) \to \nu_{\varepsilon,M}$.

$$\nu_{\varepsilon,M} \sim \psi(t) = F(B).$$

We have the following parabolic "SPDE":

$$\partial_t \psi = \Delta_{\Lambda_{\varepsilon,M}} \psi - (m^2 + a_\varepsilon) \psi - \lambda \psi^3 + \xi_\varepsilon, \qquad t \geqslant 0, x \in \Lambda_{\varepsilon,M}$$

where $\xi_\varepsilon = \xi_\varepsilon(t, x) = \partial_t B(t, x)$ is a spatial discretization of the space-time white noise. We have to choose $a_\varepsilon \to +\infty$ in the right way otherwise **no limit**.

We want to rely on PDE methods (not even Markov process/Dirichlet form methods). The idea is to get estimates on ψ which depends only on ξ:

$$\|\psi\| \lesssim F(\|\xi\|)$$

The problems I need to overcome are the following

- **The $\varepsilon \to 0$ limit and renormalization**: this will follow from PDE arguments with some very small probabilistic component (in very specific points). This mixing is allowed by the ideas coming from *rough path theory* and *regularity structures* or *paracontrolled distribution*. This gives control of the $\varepsilon \to 0$ limit (locally in space-time). This is the key to tightness in the UV limit.

- **The $M \to \infty$ limit and global in time solutions:** this is purely PDE argument which uses the good sign of $\lambda > 0$ ("coming down from infinity"). This will allow to have local bounds of ψ wrt. the size of the noise (locally indeed). This is the key to tightness in the infinite volume limit.

Aim is to get tightness, this will follow from deterministic estimates of the from $\|\psi(t)\| \lesssim F(\|\xi\|)$ since we can then just say we aim for estimates of the form

$$\int_{\mathscr{S}'(\mathbb{R}^3)} \|\rho\varphi\|^p_{H^{-1/2-\kappa}(\mathbb{R}^3)} \nu^\lambda_{\varepsilon,M}(\mathrm{d}\varphi) < \infty$$

unifromly in ε, M for a smooth weight $\rho: \mathbb{R}^3 \to \mathbb{R}_+$ of the form

$$\rho(x) = (1 + h|x|^2)^{-\nu}$$

for some $h > 0$ and $\nu > 0$ and some $\kappa > 0$. Recall that the regularity of the Gaussian free field (i.e. $\lambda = 0$, with covariance $(m^2 - \Delta)^{-1}(x, y) \approx |x - y|^{-1}$) is only in scale $-1/2 - \kappa$ for some positive κ and one can take the GFF having values in any space of the form $B^{-1/2-\kappa}_{p,q}$ and does not make much difference which integrability you look at so at the end of the day just take $p, q = \infty, \infty$ and we let $\mathscr{C}^\alpha = B^\alpha_{\infty,\infty}$ and $H^\alpha = B^\alpha_{2,2}$ (usual Sobolev spaces).

Then with estimates of the form

$$\|\psi(T)\| \lesssim F_T(\|\xi\|)$$

we could say (recall that we take $\psi(0) \sim \nu_{\varepsilon,M}$)

$$\int_{\mathscr{S}'(\mathbb{R}^3)} \|\rho\varphi\|^p_{H^{-1/2-\kappa}(\mathbb{R}^3)} \nu^\lambda_{\varepsilon,M}(\mathrm{d}\varphi) = \mathbb{E}[\|\rho\psi(0)\|^p_{H^{-1/2-\kappa}(\mathbb{R}^3)}]$$

$$= \mathbb{E}[\|\rho\psi(T)\|^p_{H^{-1/2-\kappa}(\mathbb{R}^3)}] \lesssim \mathbb{E}[F(\|\xi\|)^p].$$

The subtle point in this idea is that the bound $F(\|\xi\|)$ better not depend on $\psi(0)$ because in general one has only

$$\|\psi(T)\| \lesssim F'(\|\psi(0)\|, \|\xi\|).$$

The time T can be arbitrarily small but has to be nonzero, if $\lambda > 0$ and then in this case we can just forget about the initial condition in the estimate (coming down from infinity).

Coming down from infinity

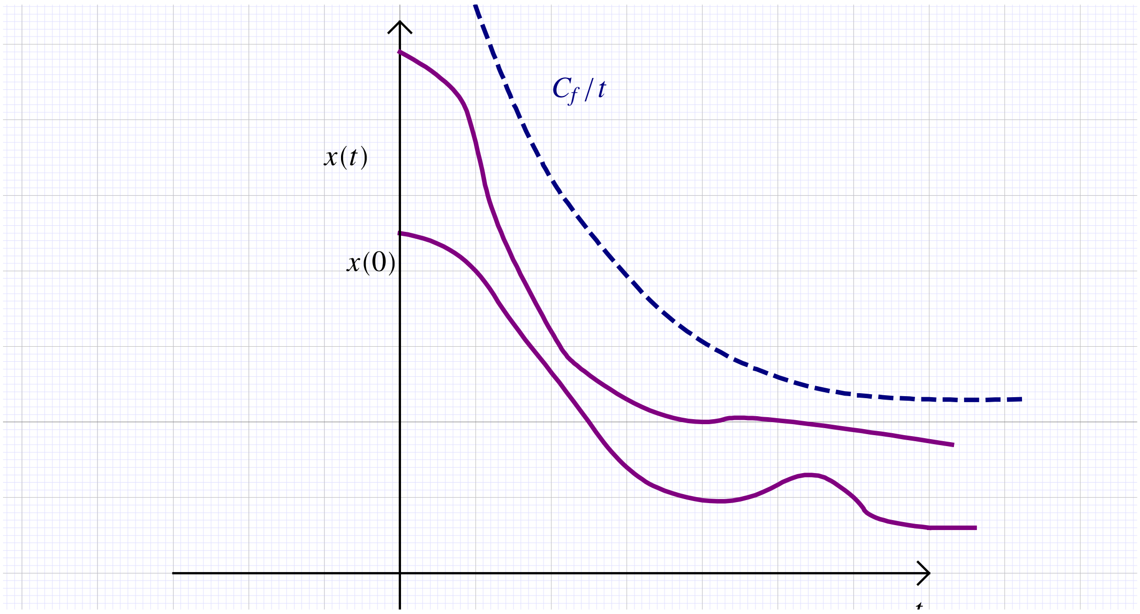
Model problem: ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\lambda x(t)^3 + f(t)$$

The idea is that any solution of this equation has the bound

$$|x(t)| \lesssim C_f t^{-1}$$

independently of the initial condition. Here by comparison the bound C_f can be taken depending only on $\|f\|_\infty$.



Maybe a proof: recall $\lambda > 0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(tx(t))^2 = tx(t)^2 + t^2 x(t)(-\lambda x(t)^3 + f(t))$$

$$= \underbrace{tx(t)^2}_{\leqslant \delta t^2 x(t)^4 + \frac{1}{\delta}} - \lambda t^2 x(t)^4 + \underbrace{t^2 x(t) f(t)}_{=(t^{1/2}x)(t^{3/2}f) \leqslant \gamma (t^{1/2}x)^4 + \frac{1}{\gamma}(t^{3/2}f)^{4/3}}$$

$$\leqslant -\lambda t^2 x(t)^4 + \delta t^2 x(t)^4 + \frac{1}{\delta} + \gamma (t^{1/2}x)^4 + \frac{1}{\gamma}(t^{3/2}f)^{4/3}$$

$$\leqslant \underbrace{(\delta + \gamma - \lambda)}_{<0} t^2 x(t)^4 + \frac{1}{\delta} + \frac{1}{\gamma}(t^{3/2}f)^{4/3}$$

$$\leqslant \frac{1}{\delta} + \frac{1}{\gamma}(t^{3/2}f)^{4/3}$$

and we obtained that

$$(tx(t))^2 \leqslant (0x(0))^2 + \int_0^t \frac{1}{\delta} + \frac{1}{\gamma}(t^{3/2}f(s))^{4/3}\mathrm{d}s \leqslant \int_0^t \frac{1}{\delta} + \frac{1}{\gamma}(s^{3/2}f(s))^{4/3}\mathrm{d}s$$

which is what I promised. This is the key to uniform global space-time estimates. It is enough just to take *some* time t for example $t = 1$.

This argument can be extended in space-time giving energy estimates for the equation

$$\partial_t \psi = \Delta_{\Lambda_{\varepsilon,M}} \psi - (m^2 + a_\varepsilon)\psi - \lambda \psi^3 + \xi_\varepsilon,$$

one considers the quantity $\|t\rho\psi\|_{L^2}^2$ and differentiate in time

$$\frac{1}{2}\partial_t \|t\rho\psi\|_{L^2}^2 = t\|\rho\psi\|_{L^2}^2 + \langle t^2\rho^2\psi, \partial_t\psi\rangle$$

$$= t\|\rho\psi\|_{L^2}^2 + \langle \rho^2\psi, \Delta_\varepsilon\psi - (m^2 + a_\varepsilon)\psi - \lambda\psi^3 + \xi_\varepsilon\rangle$$

$$= \underbrace{-\|t\rho\nabla_\varepsilon\psi\|_{L^2}^2 - m^2\|t\rho\psi\|_{L^2}^2 - \lambda t^2\|\rho^{1/2}\psi\|_{L^4}^4}_{\text{good guys}} + \underbrace{t^2\langle \rho^2\psi, \xi_\varepsilon\rangle + t\|\rho\psi\|_{L^2}^2}_{\text{bad guys}}$$

For example we can control $t\|\rho\psi\|_{L^2}^2 \leqslant \delta t^2\|\rho^{1/2}\psi\|_{L^4}^4 + C_{\rho,\delta}$ and the other terms similarly. The good guys tell me that I have control of some weigthed H^1, L^4 norm of the solution.

One can use careful chosen spatial weight which are of compact support (like in the time direction) (see the papers of Moinat–Weber).

This idea works as long as we keep $\varepsilon > 0$. Because essentially there is no issue of spatial regularity even if one has to take care of the fact that ξ_ε is not really a nice function in time (one has to use Ito formula to get this energy estimate). I don't continue on this line because anyway we will have to change our argument to ha ve estimates uniform in $\varepsilon > 0$.

The problem is that as $\varepsilon \to 0$ we cannot hope that ψ lives in H^1 for example, actually we “know” that ψ has to live in $H^{-1/2-\kappa}$.

Paracontrolled expansion

We need not a strategy for taking the $\varepsilon \to 0$ limit uniformly. The basic idea is that I want to get rid of the irregular driving term ξ_ε. We do so by subtractions: the picture is that on small spatial scales the solution behaves like a perturbation of the linear equation

I introduce X_ε which is the stationary solution of the linear eq

$$\partial_t X_\varepsilon = \Delta_\varepsilon X_\varepsilon - m^2 X_\varepsilon + \xi_\varepsilon.$$

This random field is such that $X_\varepsilon(t)$ is the GFF, in particular

$$X_\varepsilon \in C(\mathbb{R}_+; \mathscr{C}^{-1/2-\kappa}(\rho^\varepsilon)),$$

indeed one has estimates of the form

$$|\Delta_i X_\varepsilon(x)| \lesssim C(\omega)(1 + \log(|x|))$$

so we need some weight in space to have the sup finite.

$$\|X_\varepsilon\| = \sup_{i\geqslant 1} (2^i)^{-1/2-\kappa} \sup_{x\in\mathbb{R}^3} (1+|x|)^{-\varepsilon} |\Delta_i X_\varepsilon(x)| < \infty$$

almost surely (this is the interface between probability and analysis).

Then we define $\psi = X_\varepsilon + \phi_\varepsilon$ and observe that letting $\mathscr{L}_\varepsilon = \partial_t - \Delta_\varepsilon + m^2$

$$\mathscr{L}_\varepsilon \phi_\varepsilon = -\lambda \phi_\varepsilon^3 - \lambda X_\varepsilon^3 + a_\varepsilon X_\varepsilon - (3\lambda X_\varepsilon^2 - a_\varepsilon)\phi_\varepsilon - 3\lambda X_\varepsilon \phi_\varepsilon^2$$

The point now is to observe that we can write $a_\varepsilon = 3\lambda a'_\varepsilon + b_\varepsilon$ so that we can define

$$\mathbb{X}_\varepsilon^3 := X_\varepsilon^3 - 3a'_\varepsilon X_\varepsilon, \qquad \mathbb{X}_\varepsilon^2 := X_\varepsilon^2 - a'_\varepsilon$$

and the equation takes the form

$$\mathscr{L}_\varepsilon \phi_\varepsilon = -\lambda \phi_\varepsilon^3 - \lambda \mathbb{X}_\varepsilon^3 - 3\lambda \mathbb{X}_\varepsilon^2 \phi_\varepsilon - 3\lambda X_\varepsilon \phi_\varepsilon^2 + b_\varepsilon (X_\varepsilon + \phi_\varepsilon)$$

with a'_ε is chosen so that $\mathbb{E}[\mathbb{X}_\varepsilon^2] = 0$, can be chosen constant in space time and $a'_\varepsilon \approx \varepsilon^{-1}$. This is the first order renormalization one has to perform. After this renormalization the random fields $\mathbb{X}_\varepsilon^3$ and $\mathbb{X}_\varepsilon^2$ have limits as $\varepsilon \to 0$ and the respective regularities are (this needs a *probabilistic* proof)

$$\mathbb{X}_\varepsilon^2 \to \mathbb{X}^2 \in C(\mathbb{R}_+; \mathscr{C}^{-1-\kappa}(\rho^\varepsilon)),$$

$$\mathbb{X}_\varepsilon^3 \to \mathbb{X}^3 \in \cap_{\delta>0} C^{-\delta}(\mathbb{R}_+; \mathscr{C}^{-3/2-\kappa-2\delta}(\rho^\varepsilon)),$$

the cube is only a space-time distribution of space-time parabolic regularity $-3/2-\kappa$. This is clear by looking at

$$\mathbb{E}\left\{\left(\int_{\mathbb{R}^3} \mathbb{X}_\varepsilon^3(t,x)\varphi(x)\mathrm{d}x\right)^2\right\} \approx \log(\varepsilon^{-1})$$

but

$$\mathbb{E}\left\{\left(\int_{\mathbb{R}_+\times\mathbb{R}^3} \mathbb{X}_\varepsilon^3(t,x)\varphi(t,x)\mathrm{d}t\mathrm{d}x\right)^2\right\} < \infty.$$

By parabolic regularity we can hope that $\phi \in \mathscr{C}^{1/2-\kappa}$, this is still not good because does not allow to control the terms

$$\mathbb{X}_\varepsilon^2 \phi_\varepsilon, \qquad 3\lambda X_\varepsilon \phi_\varepsilon^2$$

since $\mathbb{X}_\varepsilon^2 \in \mathscr{C}^{-1-\kappa}$ and $X_\varepsilon \in \mathscr{C}^{-1/2-\kappa}$ and the products between $\mathscr{C}^\alpha$ and $\mathscr{C}^\beta$ are well defined only if $\alpha + \beta > 0$.

We can define this

$$\mathscr{L}_\varepsilon X_\varepsilon^{\Psi} = -\lambda \mathbb{X}_\varepsilon^3$$

and then redefine

$$\psi = X_\varepsilon + \underbrace{X_\varepsilon^{\Psi} + \zeta_\varepsilon}_{\phi_\varepsilon}$$

to see that now

$$\mathscr{L}_\varepsilon \zeta_\varepsilon = -\lambda \phi_\varepsilon^3 - 3\lambda \mathbb{X}_\varepsilon^2 \phi_\varepsilon - 3\lambda X_\varepsilon \phi_\varepsilon^2 + b_\varepsilon (X_\varepsilon + \phi_\varepsilon)$$

Now we know that $\zeta_\varepsilon \in \mathscr{C}^{1-\kappa}$ so $\phi_\varepsilon = X_\varepsilon^{\Psi} + \zeta_\varepsilon$ so

$$X_\varepsilon \phi_\varepsilon^2 = X_\varepsilon (X_\varepsilon^{\Psi} + \zeta_\varepsilon)^2 = X_\varepsilon (X_\varepsilon^{\Psi})^2 + \underbrace{2 X_\varepsilon X_\varepsilon^{\Psi} * \zeta_\varepsilon + X_\varepsilon * \zeta_\varepsilon^2}_{\text{well defined since } (-1/2-\kappa)+(1-\kappa)>0}$$

The idea here is to use again probability to show that $X_\varepsilon (X_\varepsilon^{\Psi})^2$ and $X_\varepsilon X_\varepsilon^{\Psi}$ are well defined in the limit.

$$X^{\Psi\!\vee} = \lim_\varepsilon X_\varepsilon X_\varepsilon^{\Psi},$$

and similar. No further renormalization neeeded. Easy.

The tricky term is really

$$\mathbb{X}_\varepsilon^2 \phi_\varepsilon = \mathbb{X}_\varepsilon^2 (X_\varepsilon^{\Psi} + \zeta_\varepsilon) = \mathbb{X}_\varepsilon^2 X_\varepsilon^{\Psi} + \mathbb{X}_\varepsilon^2 \zeta_\varepsilon$$

the product $\mathbb{X}_\varepsilon^2 \zeta_\varepsilon$ is not defined!!!! Model problem:

$$\mathscr{L}_\varepsilon \zeta_\varepsilon = -\mathbb{X}_\varepsilon^2 \zeta_\varepsilon + b_\varepsilon (X_\varepsilon + \phi_\varepsilon) + \cdots$$

And this is serious problem… What I explained up to know was clear from Da Prato–Debussche (2005) and it took until Hairer (2013) to find a solution. The idea (in a different language) is to have a good model of how ζ_ε looks like.

Now the idea is to say that "at first order" ζ_ε "looks like" the solution $X_\varepsilon^{\curlyvee} \in \mathscr{C}^{1-\kappa}$ of the equation

$$\mathscr{L}_\varepsilon X_\varepsilon^{\curlyvee} = \mathbb{X}_\varepsilon^2.$$

$$\zeta_\varepsilon = A_\varepsilon (X_\varepsilon^{\curlyvee}) + r_\varepsilon$$

with $r_\varepsilon \in \mathscr{C}^{3/2-\kappa}$ and A_ε a linear operator such that we can write

$$\mathbb{X}_\varepsilon^2 \zeta_\varepsilon + b_\varepsilon (X_\varepsilon + \phi_\varepsilon) = \mathbb{X}_\varepsilon^2 A_\varepsilon (X_\varepsilon^{\curlyvee}) + b_\varepsilon (X_\varepsilon + \phi_\varepsilon) + \underbrace{\mathbb{X}_\varepsilon^2 r_\varepsilon}_{\text{OK}} = B_\varepsilon (\mathbb{X}_\varepsilon^2 X_\varepsilon^{\curlyvee}) + b_\varepsilon (X_\varepsilon + \phi_\varepsilon) + \cdots$$

$$= B_\varepsilon (\mathbb{X}_\varepsilon^2 X_\varepsilon^{\curlyvee} - b_\varepsilon) + \cdots$$

and one can prove that $\mathbb{X}_\varepsilon^2 X_\varepsilon^{\curlyvee}$ is well defined provided is renormalized. But then one can choose b_ε to renormalize it.

$$\psi = \underbrace{X_\varepsilon}_{-1/2-\kappa} + \underbrace{X_\varepsilon^{\Psi}}_{-1-\kappa} + \underbrace{A_\varepsilon (X_\varepsilon^{\curlyvee})}_{1-\kappa} + \underbrace{r_\varepsilon}_{3/2-\kappa}$$