The scaling limit of non-integrable Ising models

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Based on joint works with

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Main reference: arXiv:2006.04458

Mathematical Methods in Field Theory and Quantum Mechanics

July 8, 2020

Outline

The model

Let $\Omega \subset \mathbb{R}^2$, $a =$ lattice spacing, $\Omega_a := a\mathbb{Z}^2 \cap \Omega$.
Our main results concern: $\Omega = \mathbb{T}_\ell$ or $\Omega = \mathcal{C}_{\ell_1,\ell_2}$.

$$H(\sigma) = -J \sum_{\langle x,y \rangle} \sigma_x \sigma_y - \lambda \sum_{X \subset \Omega_a} V(X) \sigma_X$$

where:

- $\langle x, y \rangle$ indicates n.n. sites in Ω_a
- $\sigma_x = \pm 1$ and $\sigma_X = \prod_{z \in X} \sigma_z$
- $V(X)$ finite-range, transl. & rot. inv., even.
- B.c.^{s}: periodic (in $\mathbb{T}_\ell$), open/periodic (in $\mathcal{C}_{\ell_1,\ell_2}$)

Gibbs measure:

$$\mu^\lambda_{\beta;a,\Omega}(A) = \frac{1}{Z^\lambda_{\beta;a,\Omega}} \sum_\sigma e^{-\beta H(\sigma)} A(\sigma).$$

Main results

$$\text{Let} \quad \varepsilon_j(x) = a^{-1}(\sigma_x \sigma_{x+a\hat{e}_j} - \mu^{\lambda}_{\beta_c;a,\Omega}(\sigma_x \sigma_{x+a\hat{e}_j})).$$

For λ small enough, there exists $\beta_c = \beta_c(\lambda)$ s.t.

$$\lim_{a\to 0} \lim_{\ell\to\infty} \mu^{\lambda}_{\beta_c;a,\mathbb{T}_\ell}(\varepsilon_{j_1}(x_1)\cdots\varepsilon_{j_n}(x_n)) = \Big(-\frac{Z}{\pi}\Big)^n |\mathrm{Pf}\Big(\frac{\mathbb{1}_{i\neq j}}{z_i - z_j}\Big)|^2$$

$$\lim_{a\to 0} \mu^{\lambda}_{\beta_c;a,\mathcal{C}_{\ell_1,\ell_2}}(\varepsilon_{j_1}(x_1)\cdots\varepsilon_{j_n}(x_n)) = \Big(-\frac{Z}{\pi}\Big)^n \mathrm{Pf}\Big[\mathbb{1}_{i\neq j} g^{\mathrm{scal}}(x_i, x_j)\Big],$$

where: $Z = Z(\lambda) = 1 + O(\lambda)$ *is analytic in* λ *and* $g^{scal}(x,y)$ *is a sum over images of*

$$f(x,y) = \frac{1}{|x-y|^2}\begin{pmatrix} x_1 - y_1 & x_2 - y_2 \\ x_2 - y_2 & -x_1 + y_1 \end{pmatrix}.$$

Outline

Main steps

1. Nearest-neighbor Ising $\leftrightarrow$ Gaussian Grassmann integral $\equiv$ free fermions
2. Non-integrable Ising $\equiv$ non-Gaussian Grassmann integral ('fermionic ϕ_2^4 theory')
3. Integrate out 'massive' degrees of freedom
4. Multiscale integration of the 'massless' d.o.f.:
   - scaling dimensions (quartic interaction: irrelevant),
   - localization and renormalization,
   - control flow of running coupling constants via counterterms,
   - dimensional gains from bulk/edge decomposition, ...

Step 1: n.n. Ising and dimers

Let $Z^0 = Z^0_{\beta;a,\Omega}$. If $\mathcal{F}_{\Omega_a}$ is Fisher lattice on Ω_a,

$$Z^0 = \text{dimers on } \mathcal{F}_{\Omega_a} = \text{Pf}(K), \qquad \text{with:}$$

- K Kasteleyn matrix (complex adjacency matrix) on $\mathcal{F}_{\Omega_a}$
- $\text{Pf}(K)$ the Pfaffian of K, where:

$$\text{Pf}(A) = \frac{1}{2^n n!}\sum_\pi (-1)^\pi A_{\pi(1)\pi(2)} \cdots A_{\pi(2n-1)\pi(2n)} = \pm\sqrt{\det A}$$

for any $2n \times 2n$ anti-symmetric matrix A.

Convenient way of representing $\text{Pf}(A)$: in terms of a Gaussian Grassmann integral.

Advantage: easy to compute minors of K^{-1} in terms of 'fermionic Wick rule' (very convenient for computing non-Gaussian perturbations thereof)

Step 1: Grassmann variables

Finite label set I with $|I| = 2n$. Grassmann variables: $\psi = \{\psi_\alpha\}_{\alpha \in I}$ s.t. $\{\psi_\alpha, \psi_\beta\} = 0$.

Grassmann algebra $\equiv$ set of Grassmann polynomials

$$Q(\psi) = c_0 + \sum_{1 \le i \le 2n} c_{1;i} \psi_i + \sum_{1 \le i < j \le 2n} c_{2;ij} \psi_i \psi_j + \cdots + c_{2n} \psi_1 \cdots \psi_{2n}$$

Analytic function of $\psi \equiv$ (truncated) Taylor series, e.g.,

$$e^{c\psi_1\psi_2} = 1 + c\psi_1\psi_2$$

Grassmann integration:

$$\int D\psi \; Q(\psi) = c_{2n}$$

Step 1: Gaussian Grassmann integration

$$\int D\psi \; e^{\frac{1}{2}\sum_{i,j}\psi_i A_{ij}\psi_j} \equiv \int D\psi \; e^{\frac{1}{2}(\psi, A\psi)} = \mathrm{Pf}(A)$$

$$\frac{1}{\mathrm{Pf}A}\int D\psi \; e^{\frac{1}{2}(\psi, A\psi)}\psi_\alpha\psi_\beta = (A^{-1})_{\alpha\beta}$$

$$\frac{1}{\mathrm{Pf}A}\int D\psi \; e^{\frac{1}{2}(\psi, A\psi)}\psi_{\alpha_1}\cdots\psi_{\alpha_k} =$$

$$= \sum_{\text{pairings } p} (-1)^p \prod_{(\alpha\beta)\in p} (A^{-1})_{\alpha\beta} = \mathrm{Pf}(G_{\alpha_1\cdots\alpha_k})$$

where $G_{\alpha_1\cdots\alpha_k}$ is the minor of A^{-1} consisting of the rows & columns labelled $\alpha_1, \ldots, \alpha_k$.

Step 1: n.n. Ising $\leftrightarrow$ Gaussian Grassmann integral

Putting things together:

$$Z^0 = \text{dimers on } \mathcal{F}_{\Omega_a} = \text{Pf}(K) = \int D\psi e^{\frac{1}{2}(\psi, K\psi)}.$$

After some algebra (block diagonalization of K):

$$Z^0 = \mathcal{N}_0 \int D\phi\, D\chi e^{\frac{1}{2}(\phi, A_c\phi)+\frac{1}{2}(\chi, A_m\chi)},$$

where $\phi = \{\phi_{x,\omega}\}^{\omega=\pm}_{x\in\Omega_a}$ and $\chi = \{\chi_{x,\omega}\}^{\omega=\pm}_{x\in\Omega_a}$.

On the torus or cylinder, A_c, A_m can be diagonalized explicitly. Key facts:

- A_m is non-singular, uniformly in β and a;
- A_c becomes singular at $\beta = \beta_c^0$ as $a \to 0$.

Step 1: structure of the free critical theory

The 'critical' propagator is:

$$g_{\omega\omega'}(x,y) := \frac{1}{\mathrm{Pf}(A_c)} \int D\phi\, e^{\frac{1}{2}(\phi, A_c \phi)} \phi_{x,\omega}\phi_{y,\omega'} \xrightarrow[a\to 0]{\beta\to\beta_c^0} g^{\mathrm{scal}}_{\omega\omega'}(x,y)$$

which behaves like

$$g^{\mathrm{scal}}(x,y) \sim \frac{1}{\pi^2}\frac{1}{|x-y|^2}\begin{pmatrix} x_1 - y_1 & x_2 - y_2 \\ x_2 - y_2 & -x_1 + y_1 \end{pmatrix}$$

for x, y 'well inside' the cylinder.

Step 2: Interacting model $\leftrightarrow$ interacting fermions

In the presence of a weak interaction, we use cluster expansion to get, for any β (eventually, we'll need $\beta = \beta_c(\lambda)$)

$$Z^\lambda = \mathcal{N}_\lambda \int D\phi \, D\chi \, e^{\frac{1}{2}(\phi, A_c \phi) + \frac{1}{2}(\chi, A_m \chi)} e^{\mathcal{V}(\phi,\chi)}.$$

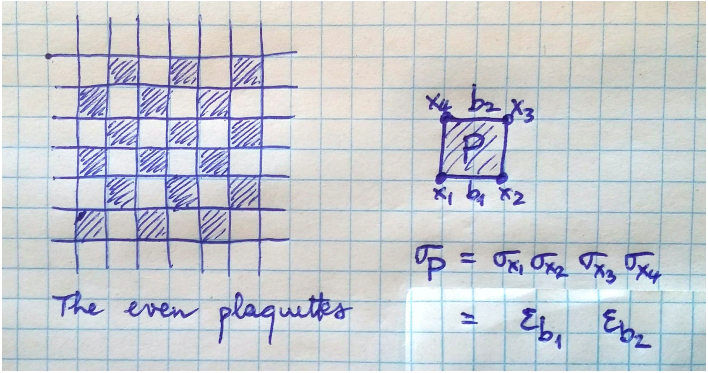
At finite a this is an identity between polynomials. $\mathcal{V}$ is of size λ and it is quartic + sextic + $\cdots$

This formula is simplest to understand in the case of the Ising model with 'plaquette interaction', in which case $\mathcal{V}$ is exactly quartic.

Step 2: Ising model with plaquette interaction

Let $a = 1$. If $b = (x, y)$ and $\varepsilon_b = \sigma_x \sigma_y$:

$$H(\sigma) = -J \sum_b \varepsilon_b - \lambda \sum_{\text{even } P} \sigma_P$$



Step 2: Ising model with plaquette interaction

$$Z^{\lambda} = \sum_{\sigma} e^{\beta J \sum_b \varepsilon_b + \lambda \sum_P \varepsilon_{b_1} \varepsilon_{b_2}}$$

$$= \sum_{\sigma} e^{\beta J \sum_b \varepsilon_b} \prod_P (\cosh \lambda + \sinh \lambda \, \varepsilon_{b_1} \varepsilon_{b_2})$$

$$= (\cosh \lambda)^{N_P} \prod_P (1 + \tanh \lambda \frac{\partial^2}{\partial x_{b_1} \partial x_{b_2}}) \sum_{\sigma} e^{\sum_b x_b \varepsilon_b} \Big|_{x_b \equiv \beta J}$$

$$= (\cosh \lambda)^{N_P} \prod_P (1 + \tanh \lambda \frac{\partial^2}{\partial x_{b_1} \partial x_{b_2}}) \int D\psi \, e^{\frac{1}{2}(\psi, K(\mathbf{x})\psi)} \Big|_{x_b \equiv \beta J}$$

$$= (\cosh \lambda)^{N_P} \int D\psi \, e^{\frac{1}{2}(\psi, K\psi)} \prod_P \left(1 + \alpha(\lambda)\psi(P)\right),$$

where $\alpha(\lambda) = \tanh \lambda \, (1 - \tanh(\beta J))^2$ and $\psi(P) =$ product of Grassmann fields at vertices of b_1, b_2.

Step 2: Ising model with plaquette interaction

Note that $1 + \alpha(\lambda)\psi(P) = e^{\alpha(\lambda)\psi(P)}$, so that, for the Ising model with plaquette interaction,

$$Z^{\lambda} = (\cosh \lambda)^{N_P} \int D\psi \; e^{\frac{1}{2}(\psi, K\psi) + \alpha(\lambda) \sum_P \psi(P)},$$

with $\psi(P)$ quartic in the Grassmann variables.

A similar strategy allows one to rewrite a general even Ising interaction as

$$Z^{\lambda} = \widetilde{\mathcal{N}}_{\lambda} \int D\psi \; e^{\frac{1}{2}(\psi, K\psi) + \mathcal{V}(\psi)},$$

with $\mathcal{V}(\psi)$ = quartic + sextic + ... and:
quartic = $O(\lambda)$, sextic = $O(\lambda^2)$, etc.

Step 2: Interacting Ising $\leftrightarrow$ non-Gaussian Grassmann int.

$$Z^\lambda = \widetilde{\mathcal{N}}_\lambda \int D\psi \; e^{\frac{1}{2}(\psi, K\psi) + \mathcal{V}(\psi)},$$

with $\mathcal{V}(\psi) =$ quartic + sextic + ... and:
quartic $= O(\lambda)$, sextic $= O(\lambda^2)$, etc.

How do we compute this non-Gaussian integral?
How do we fix the right critical temperature?

The model looks like a ϕ^4_2 model: we will employ multiscale methods borrowed from the RG theory of ϕ^4_3 and ϕ^4_4, due to: Glimm, Jaffe, Spencer, Benfatto, Gallavotti, Magnen, Feldman, Magnen, Rivasseau, Gawedzki, Kupiainen, ...

Step 2: critical+massive variables for Grassmann integral

Via the same block diagonalization of K used above,

$$Z^\lambda = \mathcal{N}_\lambda \int D\phi\, D\chi\, e^{\frac{1}{2}(\phi, A_c\phi) + \frac{1}{2}(\chi, A_m\chi) + \mathcal{V}(\phi,\chi)}.$$

We denote by $P_c(D\phi) = \frac{1}{\mathrm{Pf}(A_c)} D\phi\, e^{\frac{1}{2}(\phi, A_c\phi)}$ and $P_m(D\chi) = \frac{1}{\mathrm{Pf}(A_m)} D\phi\, e^{\frac{1}{2}(\phi, A_m\phi)}$, so that

$$Z^\lambda = e^{E(\lambda)} \int P_c(D\phi)\, P_m(D\chi) e^{\mathcal{V}(\phi,\chi)}.$$

Now, since χ is massive, we integrate it out first.

Step 3: integration of the massive variables

We compute the integral over χ and let:

$$e_1(\lambda) + \mathcal{V}^{(0)}(\phi) = \log \int P_m(d\chi) e^{\mathcal{V}(\phi,\chi)}$$

'Standard' cluster expansion for Grassmann integrals (Gawedzki-Kupiainen, Lesniewski, Battle-Brydges--Federbush-Kennedy, Benfatto-Gallavotti, Feldman--Magnen-Rivasseau-Trubowitz, ...) implies that:

- $|\Omega_a|^{-1} e_1(\lambda)$ is analytic in λ, for λ small enough, uniformly in Ω_a;
- the kernels of $\mathcal{V}^{(0)}(\phi)$ are analytic in λ, uniformly in Ω_a, and decay exp. on scale $\sim a = 1$.

Step 4: multiscale integration of the critical variables

We are left with

$$Z^\lambda = e^{E_0(\lambda)} \int P_c(D\phi) e^{\mathcal{V}^{(0)}(\phi)},$$

where $E_0(\lambda) = E(\lambda) + e_1(\lambda)$ and $\phi = \{\phi_{x,\omega}\}_{x\in\Omega_a}^{\omega=\pm}$.

A naive attempt of applying cluster expansion for

$$e_0(\lambda) = \log \int P_c(d\phi) e^{\mathcal{V}^{(0)}(\phi)}$$

fails, due to slow decay of the off-diagonal elements of $A_c^{-1}(x,y) \simeq g^{\mathrm{scal}}(x,y) \sim |x-y|^{-1}$ at large separation $|x-y|$.

Step 4: counterterms

Apparent divergence of naive perturbation theory indicates that the interaction tends to modify quantitatively the reference Gaussian 'measure' $P_c(d\phi)$. We expect that:

1. the interaction shifts the critical temperature,
2. asymptotically (at large separation) modifies the covariance of ϕ by a multiplicative factor.

In order to take this into account, we let $\beta = = \beta_c^0 + \nu(\lambda) \equiv \beta_c(\lambda)$, with $\nu(\lambda) = O(\lambda)$, and we rescale ϕ by a factor $1 + \zeta(\lambda)$, with $\zeta(\lambda) = O(\lambda)$. In order to obtain a convergent perturbative expansion, $\nu = \nu(\lambda)$ and $\zeta = \zeta(\lambda)$ must be fixed appropriately.

Step 4: counterterms

More precisely, in $P_c(D\phi) \propto D\phi\, e^{\frac{1}{2}(\phi, A_c\phi)}$ we rewrite:

$$(\phi, A_c\phi) = \mathcal{Z}(\phi, A_c^0\phi) + \text{difference},$$

with $\mathcal{Z} = 1 + \zeta$ and $A_c^0 = A_c\big|_{\beta=\beta_c^0}$

A straightforward computation shows that

$$P_c(D\phi) \propto P_{\le 0}(D\phi) e^{\nu_0(\phi,\phi)+\zeta_0(\phi,\partial\phi)}, \qquad \text{with:}$$

- $P_{\le 0}(D\phi) \propto D\phi\, e^{\frac{\mathcal{Z}}{2}(\phi, A_c^0\phi)}$
- ν_0, ζ_0 explicit functions of ν, ζ of $O(\lambda)$
- (ϕ, ϕ) is a shorthand for $\sum_z \phi_{z,+}\phi_{z,-}$
- $(\phi, \partial\phi)$ is a shorthand for

$$\sum_z (\phi_{z,+}, \phi_{z,-}) \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_2 & -\partial_1 \end{pmatrix} \begin{pmatrix} \phi_{z,+} \\ \phi_{z,-} \end{pmatrix}$$

Step 4: multiscale decomposition

We arrived at:

$$Z^\lambda = e^{\widetilde{E}_0(\lambda)} \int P_{\leq 0}(D\phi) e^{\mathcal{V}^{(0)}(\phi)+\nu_0(\phi,\phi)+\zeta_0(\phi,\partial\phi)}.$$

Recall that covariance, or propagator, of $P_{\leq 0}(D\phi)$ is

$$g^{(\leq 0)}(x,y) := \frac{1}{\mathcal{Z}}(A_c^0)^{-1}(x,y)$$

which decays like $1/|x-y|$ at large separation; i.e., $g^{(\leq 0)}$ is asymptotically homogeneous of degree -1.

We decompose $g^{(\leq 0)}$ as sum of exp. decaying prop.:

$$g^{(\leq 0)}(x,y) = \sum_{h\leq 0} g^{(h)}(x,y), \quad g^{(h)}(x,y) \simeq 2^h g^{(0)}(2^h x, 2^h y),$$

where $g^{(0)}$ is exp. decaying on scale $a = 1$.

Step 4: multiscale decomposition

We write:

$$g^{(\le 0)}(x,y) = g^{(0)}(x,y) + g^{(\le -1)}(x,y),$$

$$g^{(\le -1)}(x,y) = \sum_{h \le -1} g^{(h)}(x,y), \quad g^{(h)}(x,y) \simeq 2^h g^{(0)}(2^h x, 2^h y)$$

where $g^{(0)}$ is exp. decaying on scale $a = 1$. This de--composition corresponds to a 'resolution of singu--larity' as a superposition of non-singular objects.

By the addition principle for Gaussian integrals

$$\int P_{\le 0}(D\phi) e^{\widetilde{\mathcal{V}}^{(0)}(\phi)} = \int P_{\le -1}(D\phi') \int P_0(D\phi'') e^{\widetilde{\mathcal{V}}^{(0)}(\phi'+\phi'')}$$

where $\widetilde{\mathcal{V}}^{(0)}(\phi) = \mathcal{V}^{(0)}(\phi) + \nu_0(\phi,\phi) + \zeta_0(\phi, \partial\phi)$, $P_0(D\phi)$ and $P_{\le -1}(D\phi)$ have prop. $g^{(0)}$ and $g^{(\le -1)}$.

Step 4: the first step of the infrared integration

We let

$$e_0(\lambda) + \mathcal{V}^{(-1)}(\phi) = \log \int P_0(D\varphi) e^{\widetilde{\mathcal{V}}^{(0)}(\phi+\varphi)}.$$

The integration w.r.t. $P_0(D\varphi)$ is the 'same' as the one w.r.t. $P_m(D\chi)$. Again, $e_0(\lambda)$ analytic in λ, kernels of $\mathcal{V}^{(-1)}(\phi)$ analytic in λ and exp. decaying (on scale 2, rather than $a = 1$).

Next, we iterate. After finite number of steps:

$$Z^\lambda = e^{E_h(\lambda)} \int P_{\leq h}(d\phi) e^{\mathcal{V}^{(h)}(\phi)},$$

where $P_{\leq h}$ has propagator $g^{(\leq h)}(x) = \sum_{j \leq h} g^{(j)}(x)$ and $\mathcal{V}^{(h)}(\phi)$ is the effective potential on scale h.

Step 4: effective potentials

$$Z^\lambda = e^{E_h(\lambda)} \int P_{\leq h}(d\phi) e^{\mathcal{V}^{(h)}(\phi)},$$

where $P_{\leq h}$ has propagator $g^{(\leq h)}(x) = \sum_{j \leq h} g^{(j)}(x)$ and $\mathcal{V}^{(h)}(\phi)$ is the effective potential on scale h:

$$\mathcal{V}^{(h)}(\phi) = \sum_{\substack{n \geq 2 \\ \omega_1, \ldots, \omega_n}} \int dx_1 \cdots dx_n \, W^{(h)}_{n;\underline{\omega}}(x_1, \ldots, x_n) \phi_{x_1,\omega_1} \cdots \phi_{x_n,\omega_n}$$

Size of $W_{n,\underline{\omega}}$ measured by weighted L_1 norm:

$$\| W^{(h)}_n \|_{\kappa,h} = \frac{1}{|\Omega_a|} \sup_{\underline{\omega}} \sum_{\underline{x}} e^{\kappa 2^h d(\underline{x})} |W^{(h)}_{n;\underline{\omega}}(\underline{x})|$$

Step 4: relevant, marginal, irrelevant terms

Rescaled, a-dimensional, kernel: letting $D_n = 2 - \frac{n}{2}$,

$$\mathcal{W}^{(h)}_{n;\underline{\omega}}(x_1, \dots, x_n) = 2^{-hD_n} 2^{-2h(n-1)} W^{(h-1)}_{n;\underline{\omega}}(2^{-h}x_1, \dots, 2^{-h}x_n)$$

whose size is measured in the norm $\| \cdot \|_\kappa := \| \cdot \|_{\kappa,0}$.

Multiscale integration + rescaling defines RG map

$$RG : \{\mathcal{W}^{(h)}_n\}_{n\geq 2} \to \{\mathcal{W}^{(h-1)}_n\}_{n\geq 2},$$

whose linearization is:

- contractive for $D_n < 0 \Leftrightarrow n \geq 6$ (irrelevant terms)
- expanding for $D_n > 0 \Leftrightarrow n = 2$ (relevant terms)
- neutral for $D_n = 0 \Leftrightarrow n = 4$ (marginal terms)

$D_n = 2 - \frac{n}{2}$ is the **scaling dimension**. In order for the multiscale integration to be well defined, we need to look more closely at the terms with $D_n \geq 0$.

Step 4: localization of the quadratic terms

We decompose the quadratic part of $\mathcal{V}^{(h)}(\phi)$, $n = 2$, into local (relevant + marginal) + irrelevant part:

$$\sum_{\omega,\omega'} \int dx\, dy\, W^{(h)}_{2;(\omega,\omega')}(x,y)\phi_{x,\omega}\phi_{y,\omega'} =$$

$$= 2^h \nu_h(\phi,\phi) + \zeta_h(\phi,\partial\phi) + \mathcal{R}\mathcal{V}^{(h)}_2(\phi),$$

where in the transl. inv. case ν_h, ζ_h are constants.

The non-local contribution $\mathcal{R}\mathcal{V}^{(h)}_2$ shrinks under the action of RG. As for the local part: we fix the counterterms ν_0, ζ_0 (that is, β_c and $\mathcal{Z}$) in such a way that ν_h, ζ_h tend exp. to zero as $h \to -\infty$.

Step 4: localization of the quartic terms

We decompose the quartic part of $\mathcal{V}^{(h)}(\phi)$, $n = 4$, into local (marginal) + irrelevant part:

$$\mathcal{V}_4^{(h)}(\phi) = \mathcal{L}\mathcal{V}_4^{(h)}(\phi) + \mathcal{R}\mathcal{V}_4^{(h)}(\phi),$$

where

$$\mathcal{L}\mathcal{V}_4^{(h)}(\phi) = \sum_{\omega_1,\ldots,\omega_4} \sum_x \lambda_{h;\underline{\omega}} \phi_{x,\omega_1}\phi_{x,\omega_2}\phi_{x,\omega_3}\phi_{x,\omega_4},$$

with

$$\lambda_{h;\underline{\omega}} = \sum_{y,z,w} W_{4;\underline{\omega}}^{(h)}(x,y,z,w).$$

However, $\mathcal{L}\mathcal{V}_4^{(h)} = 0$, thanks to the Grassmann anti-commuting rules, because $\omega_i = \pm$.

Step 4: conclusions (in the plane)

KEY CANCELLATION: The number of independent critical Grassmann fields (2 per site, $\phi_{x,\omega}$, with $\omega = \pm$) guarantees that the marginal quartic part vanishes. In the plane, there is only one relevant coupling, ν_h, and one marginal coupling, ζ_h, whose flow is controlled by the choice of the two counterterms NOTE: we have as many counterterms as relevant+marginal couplings).

In conclusion, by properly tuning the temperature and choosing the multiplicative renormalization $\mathcal{Z}$, perturbation theory in the plane is convergent.

Step 4: RG flow in the cylinder

What if we have a boundary? A priori ν_h, ζ_h are x_2-dependent. We decompose $\mathcal{V}_2^{(h)}$ (and, conse--quently, ν_h, ζ_h) into a bulk + edge contribution.

The boundary contribution is dimensionally better behaved than the bulk part: effectively,

- the boundary part of ν_h is **marginal**,
- the boundary part of ζ_h is **irrelevant**.

Step 4: the boundary marginal term

We are left with the boundary part of ν_h, whose local part at the boundary reads

$$\delta_h N_\partial(\phi) \equiv \delta_h \sum_{x \in \partial\Omega_a} \phi_{x,+}\phi_{x,-}$$

In general, this has the effect of changing the boundary conditions of the quadratic action.

A priori, δ_h may diverge logarithmically, or with an anomalous critical exponents, as $h \to -\infty$.

In our case, an additional cancellation, related to an approximate image rule for $g^{(h)}$, shows that $N_\partial(\phi)$ is effectively irrelevant: cylindrical b.c. are asymptotically invariant under RG flow.

Open problems

- Spin-spin correlator? It corresponds to partition function in the presence of a ‘string defect’. Possible strategy: treat string defect in a way analogous to boundary terms.
- Conformal covariance? More general domains? Our method relies too heavily on the exact solution in the cylinder. Can we perform the RG without diagonalizing the propagator?
- Effect of boundary conditions in more general models (e.g., interacting dimers, 6V, 8V)? There, we need to study the RG flow of δ_h (anomalous flow? boundary critical exponents?).

Thank you!