The scaling limit of non-integrable Ising models

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Based on joint works with

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Outline

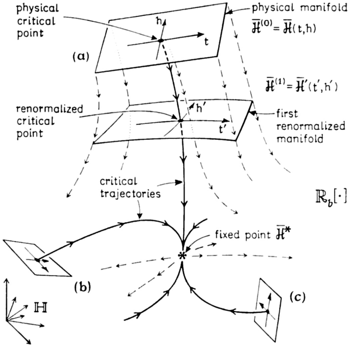
Universality, scaling limits and Renormalization Group

The scaling limit of the Gibbs measure of a critical stat-mech model is expected to be **universal**.

Conceptually, the route towards universality is clear:

1. Integrate out the small-scale d.o.f., rescale, show that the critical model reaches a fixed point (Wilsonian RG).

2. Use CFT to classify the possible fixed points (complete classification in 2D; recent progress in 3D).

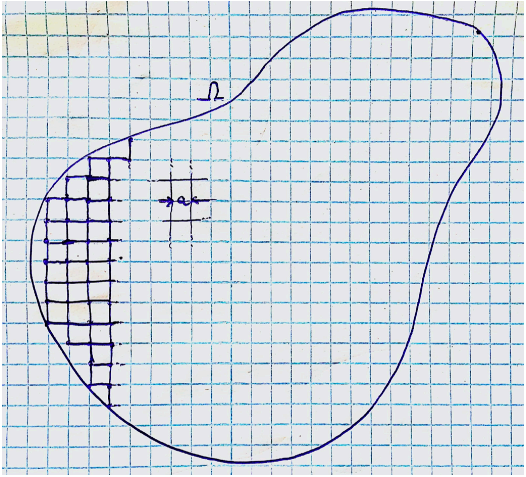


Challenge: Prove universality starting from an explicit class of microscopic Hamiltonians.

'Interacting' Ising

In this talk: review selected results on universality of non-integrable 2D models. Focus on: **Ising models**.

Let $\Omega \subset \mathbb{R}^2$, $a =$ lattice spacing, $\Omega_a := a\mathbb{Z}^2 \cap \Omega$.



‘Interacting’ Ising

Let $\Omega \subset \mathbb{R}^2$, $a =$ lattice spacing, $\Omega_a := a\mathbb{Z}^2 \cap \Omega$.

$$H(\sigma) = -J \sum_{\langle x,y \rangle} \sigma_x \sigma_y - \lambda \sum_{X \subset \Omega_a} V(X) \sigma_X$$

where:

- $\langle x, y \rangle$ indicates n.n. sites in Ω_a
- $\sigma_x = \pm 1$ and $\sigma_X = \prod_{z \in X} \sigma_z$
- $V(X)$ finite-range, transl. inv., even; λ small
- Boundary conditions: open, periodic, or $+/-$

Gibbs measure:

$$\mu^{\lambda}_{\beta;a,\Omega}(A) = \frac{1}{Z^{\lambda}_{\beta;a,\Omega}} \sum_{\sigma} e^{-\beta H(\sigma)} A(\sigma).$$

Phase transition

Take a fixed, $\Omega \nearrow \mathbb{R}^2$. Known:

- Small enough β: unique infinite volume Gibbs state with exp. decaying correlations.
- Large enough β: two distinct pure states with $\mu^{\lambda}_{\beta;a,\mathbb{R}^2}(\sigma_x) \neq 0$ & exp. decaying truncated corr.
- If $\lambda V \geq 0$, unique value of β, called $\beta_c = \beta_c(\lambda)$, separating magnetized from non-magnet. phase.

Expected: at $\beta = \beta_c$, unique infinite volume Gibbs state, denoted

$$\mu^{\lambda}_{\beta_c;a,\mathbb{R}^2} = \lim_{\Omega \nearrow \mathbb{R}^2} \mu^{\lambda}_{\beta_c;a,\Omega},$$

with polynomially decaying correlations.

Critical theory

Expected: fix $\beta = \beta_c$, fix Ω, rescale spin variables:

$$\sigma(x) = a^{-1/8}\sigma_x, \quad \varepsilon_j(x) = a^{-1}(\sigma_x\sigma_{x+a\hat{e}_j} - \mu^{\lambda}_{\beta_c;a,\Omega}(\sigma_x\sigma_{x+a\hat{e}_j}))$$

Then take $a \to 0$: the limit

$$\lim_{a\to 0} \mu^{\lambda}_{\beta_c;a,\Omega}(\sigma(x_1)\cdots\sigma(x_n)\varepsilon_{j_1}(y_1)\cdots\varepsilon_{j_m}(y_m))$$

should exist and satisfy remarkable properties. E.g.,

- Be independent of λ, up to a finite multiplicative renormalization, and equal to the correlations of the minimal CFT model with $c = 1/2$.
- Be conformal covariant under Riemann mapping of $\Omega \to \Omega'$. Correlations in any domain obtained from those in the half-plane.

Universal finite-size corrections

Fix $\beta = \beta_c$, fix Ω, compute the free energy:

$$\log Z^{\lambda}_{\beta_c;a,\Omega} = a^{-2}|\Omega| f(\lambda) + a^{-1}|\partial\Omega|\tau(\lambda) + c_\Omega(a,\lambda).$$

Here $f(\lambda)$ and $\tau(\lambda)$ independent of Ω. Expected: if χ = Euler number $\neq 0$, and $c = \frac{1}{2}$ = central charge:

$$\lim_{a\to 0} \frac{c_\Omega(a,\lambda)}{\log(a^{-1}|\partial\Omega|)} = -\frac{1}{6}c\chi;$$

if $\chi = 2 - 2h - b = 0$, then

$$\lim_{a\to 0} c_\Omega(a,\lambda) = c^0_\Omega \qquad \text{independent of } \lambda.$$

E.g., if Ω = torus with aspect ratio ξ,

$$c^0_\Omega = \log(\theta_2 + \theta_3 + \theta_4) - \frac{1}{3}\log(4\theta_2\theta_3\theta_4),$$

where $\theta_i = \theta_i(e^{-\pi\xi})$ are Jacobi theta functions.

Recent results

In recent years, constructive RG allowed us to prove some aspects of the picture above for λ small:

- for Ω plane or finite cylinder, we constructed the scaling limit of multi-point energy correlations;
- for Ω torus, we proved $c^0_\Omega \sim \frac{\pi}{6} c\xi$ as $\xi \to \infty$.

In these lectures: focus on 1^{st} result. Proof for finite cylinder very recent, it opens several perspectives:

- scaling limit of energy corr. in rectangles
- computation of c^0_Ω in finite tori and cylinders and of $\lim_{a\to 0} \frac{c_\Omega(a,\lambda)}{\log(a^{-1}|\partial\Omega|)}$ in rectangles
- ...

Outline

Nearest neighbor Ising

If $\lambda = 0$,

$$H = -J \sum_{\langle x,y \rangle} \sigma_x \sigma_y,$$

the model is exactly solvable: solution due to Onsager, Kaufman, Yang, Kasteleyn, Kac-Ward, Montroll-Potts-Ward, Schultz-Lieb-Mattis, McCoy--Wu, Tracy, Barouch, Perk, Kadanoff-Ceva, ...

E.g., fix a and send $\Omega \nearrow \mathbb{R}^2$: closed formulas for the free energy, specific heat, magnetization, energy and spin correlations (along special directions) are known.

Nearest neighbor Ising: scaling limit

At $\beta_c = \frac{\log(\sqrt{2}+1)}{2J} \equiv \beta_c^0$, polynom. decay of corr^{ns}

From explicit formulas, scaling limit in the plane:

$$\lim_{a\to 0} \mu^0_{\beta_c;a,\mathbb{R}^2}(\sigma(x)\sigma(y)) = \frac{A}{|x-y|^{1/4}}, \quad A = 0.70338016\cdots$$

$$\lim_{a\to 0} \mu^0_{\beta_c;a,\mathbb{R}^2}(\varepsilon_{j_1}(x)\varepsilon_{j_2}(y)) = \frac{1}{\pi^2}\frac{1}{|x-y|^2}.$$

More in general, the n-point energy correlations are

$$\lim_{a\to 0} \mu^0_{\beta_c;a,\mathbb{R}^2}(\varepsilon_{j_1}(x_1)\cdots\varepsilon_{j_n}(x_n)) = \pi^{-n}|\mathrm{Pf}K(z_1,\ldots,z_n)|^2,$$

where $z_j = (x_j)_1 + i(x_j)_2$ and $K_{ij}(z_1,\ldots,z_n) = \frac{\mathbb{1}_{i\neq j}}{z_i - z_j}$.

Nearest neighbor Ising: spin-spin correlations

The general formula for the scaling limit of $2n$-point spin can be guessed by CG/CFT methods, but remained elusive for many years. Recently proved by Dubedat, Chelkak-Hongler-Izyurov:

$$\lim_{a\to 0}\mu^0_{\beta_c;a,\mathbb{R}^2}(\sigma(x_1)\cdots\sigma(x_{2n})) =$$
$$= \Big[\Big(\frac{A}{\sqrt{2}}\Big)^{2n}\sum_{\substack{\varepsilon_1,\dots,\varepsilon_{2n}=\pm\\ \varepsilon_1+\cdots+\varepsilon_{2n}=0}}\prod_{1\le i<j\le 2n}|z_i-z_j|^{\varepsilon_i\varepsilon_j/2}\Big]^{1/2}.$$

Other domains: in the half-plane, similar formulas (via 'image method'). More general domains, via Riemann map (Smirnov, Chelkak-Hongler-Izyurov)

Outline

Theorem 1 (G-Greenblatt-Mastropietro 2012)

Let $V(X)$ be finite-range, rotation-symmetric and λ small enough. There exists $\beta_c = \beta_c(\lambda)$, analytic in λ, s.t., letting $\mu^{\lambda}_{\beta_c;a,\mathbb{R}^2} = \lim_{L\to\infty} \mu^{\lambda}_{\beta_c;a,\mathbb{T}^2_L}$, for any $n > 1$

$$\lim_{a\to 0} \mu^{\lambda}_{\beta_c;a,\mathbb{R}^2}(\varepsilon_{j_1}(x_1)\cdots\varepsilon_{j_n}(x_n)) = \left(\frac{Z}{\pi}\right)^n |\mathrm{Pf}K(z_1,\ldots,z_n)|^2$$

where $Z = Z(\lambda) = 1 + O(\lambda)$ is analytic in λ.

Remarks

- Main steps of proof: (i) represent. of generating function of correlations as a non-Gaussian Grassmann integral, (ii) RG computation thereof
- The RG proof provides explicit bounds on the speed of convergence of the finite-a n-point energy correlations as $a \to 0$.
- λ and V can have either signs.
- Rotational invariance unimportant.
- We also control the massive scaling limit $\beta(a) = \beta_c(\lambda) + a\frac{m_0}{4J}$. In this case, the interacting propagator has dressed mass $m = m_0(1 + O(\lambda))$.
- If $\beta \neq \beta_c(\lambda)$, proof much easier. At finite a, exp. decay with correlation length $\propto a$.

Remarks

The method of proof is (too heavily) based on translational invariance, which guarantees that:

1. quadratic part of the effective action is explicitly diagonalizable at all RG steps;
2. the effective couplings are constants, rather than functions of the position.

Important to develop RG w/o translational inv. for:

1. proving conformal covariance of the scaling limit under deformations of the domain;
2. computing boundary critical exponents (in 2D Ising, dimer, vertex models, in 1D quantum spin chains, etc.);
3. understanding critical models with defects (Kondo problem, Many Body Localization, etc.)

Interacting Ising: energy correlations in the cylinder.

Theorem 2 (Antinucci-G-Greenblatt 2020)

Same assumptions as Thm 1, same $\beta_c = \beta_c(\lambda)$ *and* $Z = Z(\lambda)$. *Domain* $\Omega =$ *cylinder of sides* ℓ_1, ℓ_2, *with p.b.c. in the horizontal direction. For any* $n > 1$,

$$\lim_{a\to 0} \mu^{\lambda}_{\beta_c;a,\Omega}(\varepsilon_{j_1}(x_1)\cdots\varepsilon_{j_n}(x_n)) = \Big(-\frac{Z}{\pi}\Big)^n \mathrm{Pf}\Big[\mathbb{1}_{i\neq j}\, g^{\mathrm{scal}}_{\omega\omega'}(x_i,x_j)\Big],$$

where $i,j \in \{1,\dots,n\}$, $\omega,\omega' \in \{\pm\}$, *and:*

$$g^{\mathrm{scal}}_{\omega\omega'}(x,y) = \sum_{n\in\mathbb{Z}^2} (-1)^n \big[f_{\omega\omega'}(x-y+\ell_n) - \omega f_{\omega\omega'}(x-\tilde y+\ell_n)\big]$$

$$\ell_n = (n_1\ell_1, 2n_2\ell_2),\ \tilde x = (x_1,-x_2),\ f(x) = \frac{1}{|x|^2}\begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}$$

Remarks

- Bounds uniform in ℓ_1, ℓ_2. Letting ℓ_1, $\ell_2 \to \infty$, we obtain the correlations in the half-plane.
- Key new ingredients, compared with Thm 1:
  (1) proof that the scaling dimension of boundary operators is better by one dimension than their bulk counterparts,
  (2) a cancellation mechanism based on an approximate image rule for the fermionic propagator allows us to control the RG flow of the marginal boundary terms.

Outline

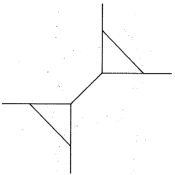
1. Introduction and overview
2. Nearest neighbor Ising: selected results
3. 'Interacting' Ising: main results
4. Sketch of the proof

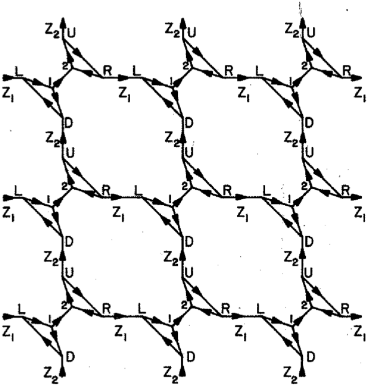
Main steps

1. Nearest-neighbor Ising $\leftrightarrow$ Gaussian Grassmann integral $\equiv$ free fermions
2. Non-integrable Ising $\equiv$ non-Gaussian Grassmann integral ('fermionic ϕ_2^4 theory')
3. Integrate out 'massive' degrees of freedom
4. Multiscale integration of the 'massless' d.o.f.:
    - scaling dimensions (quartic interaction: irrelevant),
    - localization and renormalization,
    - control flow of running coupling constants via counterterms,
    - dimensional gains from bulk/edge decomposition, ...

Well known (Fisher 1966): nearest neighbor Ising

(in high temperature contour representation: gas of polygons γ with prob. weight $(\tanh \beta J)^{|\gamma|}$) $\equiv$ dimers on 'Fisher lattice'

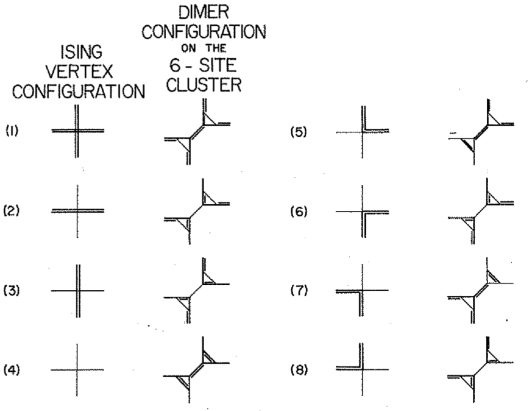




Step 1: nearest-neighbor Ising $\leftrightarrow$ dimers

Well known (Fisher 1966): nearest neighbor Ising

(in high temperature contour representation: gas of polygons γ with prob. weight $(\tanh \beta J)^{|\gamma|}$) $\equiv$ dimers on 'Fisher lattice'



Step 1: Kasteleyn's theorem

Let $Z^0 = Z^0_{\beta;a,\Omega}$. If $\mathcal{F}_{\Omega_a}$ is Fisher lattice on Ω_a,

$$Z^0 = \text{dimers on } \mathcal{F}_{\Omega_a} = \text{Pf}(K), \qquad \text{with:}$$

- K Kasteleyn matrix (complex adjacency matrix) on $\mathcal{F}_{\Omega_a}$
- $\text{Pf}(K)$ the Pfaffian of K, where:

$$\text{Pf}(A) = \frac{1}{2^n n!} \sum_{\pi} (-1)^{\pi} A_{\pi(1)\pi(2)} \cdots A_{\pi(2n-1)\pi(2n)} = \pm\sqrt{\det A}$$

for any $2n \times 2n$ anti-symmetric matrix A.

Convenient way of representing $\text{Pf}(A)$: in terms of a Gaussian Grassmann integral.

Advantage: easy to compute minors of K^{-1} in terms of 'fermionic Wick rule' (very convenient for computing non-Gaussian perturbations thereof)

Step 1: Grassmann variables

Finite label set I with $|I| = 2n$. Grassmann variables: $\psi = \{\psi_\alpha\}_{\alpha \in I}$ s.t. $\{\psi_\alpha, \psi_\beta\} = 0$.

Grassmann algebra $\equiv$ set of Grassmann polynomials

$$Q(\psi) = c_0 + \sum_{1 \le i \le 2n} c_{1;i} \psi_i + \sum_{1 \le i < j \le 2n} c_{2;ij} \psi_i \psi_j + \cdots + c_{2n} \psi_1 \cdots \psi_{2n}$$

Analytic function of $\psi \equiv$ (truncated) Taylor series, e.g.,

$$e^{c\psi_1\psi_2} = 1 + c\psi_1\psi_2$$

Grassmann integration:

$$\int D\psi \; Q(\psi) = c_{2n}$$

Step 1: Gaussian Grassmann integration

$$\int D\psi \; e^{\frac{1}{2}\sum_{i,j}\psi_i A_{ij}\psi_j} \equiv \int D\psi \; e^{\frac{1}{2}(\psi, A\psi)} = \mathrm{Pf}(A)$$

$$\frac{1}{\mathrm{Pf}A}\int D\psi \; e^{\frac{1}{2}(\psi, A\psi)}\psi_\alpha\psi_\beta = (A^{-1})_{\alpha\beta}$$

$$\frac{1}{\mathrm{Pf}A}\int D\psi \; e^{\frac{1}{2}(\psi, A\psi)}\psi_{\alpha_1}\cdots\psi_{\alpha_k} =$$

$$= \sum_{\text{pairings } p} (-1)^p \prod_{(\alpha\beta)\in p} (A^{-1})_{\alpha\beta} = \mathrm{Pf}(G_{\alpha_1\cdots\alpha_k})$$

where $G_{\alpha_1\cdots\alpha_k}$ is the minor of A^{-1} consisting of the rows & columns labelled $\alpha_1, \ldots, \alpha_k$.

Step 1: n.n. Ising $\leftrightarrow$ Gaussian Grassmann integral

Putting things together:

$$Z^0 = \text{dimers on } \mathcal{F}_{\Omega_a} = \text{Pf}(K) = \int D\psi e^{\frac{1}{2}(\psi, K\psi)}.$$

Thank you!