Ginzburg-Landau Theory of Superconductivity

[lecture 2 – part II]

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DIPARTIMENTO DI MATEMATICA

Mathematical Methods in Field Theory and Quantum Mechanics
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Outline

1. Recap:
   - Energy and density asymptotics between H_{c1} and H_{c2}.

- Surface superconductivity [Co]:
  2. Pan's conjecture and its proof [CR1–2];
  3. Curvature effects on surface superconductivity [CR2–3,CDR].
  4. Effects of boundary singularities (corners) [CG1–2].

Main References

- [Co] M. C., *arXiv:2003.00521*, *Boll. Unione Mat. Italiana* online (2020).
- [CR1] M. C., N. Rougerie, *Commun. Math. Phys.* **332** (2014).
- [CR2] M. C., N. Rougerie, *Arch. Rational Mech. Anal.* **219** (2015).
- [CR3] M. C., N. Rougerie, *Lett. Math. Phys.* **106** (2016).
- [CDR] M. C., B. Devanarayanan, N. Rougerie,, *Eur. Phys. J. B* **90** (2017).
- [CG1] M. C., E.L. Giacomelli, *Rev. Math. Phys.* **29** (2017).
- [CG2] M. C., E.L. Giacomelli, *arXiv:1908.10112* (2019).

Energy and Density Asymptotics

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$$\mathcal{E}_{\varepsilon}^{\mathrm{GL}}[\Psi, \mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r} \, \left\{ \left| \left(\nabla + i \tfrac{\mathbf{A}}{\varepsilon^2} \right) \Psi \right|^2 - \tfrac{1}{2b\varepsilon^2} \left(2|\Psi|^2 - |\Psi|^4 \right) + \tfrac{1}{\varepsilon^4} \left| \mathrm{curl} \mathbf{A} - 1 \right|^2 \right\}$$

- G.s. energy $E_{\varepsilon}^{\mathrm{GL}}$ and g.s. configuration $(\Psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ of the functional $\mathcal{F}_{\varepsilon}[\Psi, \mathbf{A}]$ in the surface superconductivity regime
$$\varepsilon \to 0, \qquad 1 < b < \Theta_0^{-1}.$$
- By the Agmon estimates, $\Psi^{\mathrm{GL}} = \mathcal{O}(\varepsilon^{\infty})$ at distances $\gg \varepsilon$ from $\partial\Omega$ and $\mathbf{A}^{\mathrm{GL}} \simeq \mathbf{F}$.
- Energy and density asymptotics:
$$E_{\varepsilon}^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} + \mathcal{O}(1), \qquad \left\| |\Psi^{\mathrm{GL}}|^2 - f_0^2\left(t\right) \right\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon).$$

Conjecture ([Pan '02])

The density $|\Psi^{\mathrm{GL}}|^2$ is close to $f_0^2(0)$ in $L^{\infty}(\partial\Omega)$.

Refined 1D Effective Energy

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$$\mathcal{E}^{1D}_{0,\alpha}[f] = \int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \tfrac{1}{2b}\left(2f^2 - f^4\right) \right\}$$

- To to refine the error term $\mathcal{O}(1)$, one has to retain the $\varepsilon-$dependent terms. This amounts to keep the corrections due to the (local) boundary curvature $k \in \mathbb{R}$: the 1D effective energy becomes

$$\mathcal{E}^{1D}_{k,\alpha}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t(1-\varepsilon k t) \left\{ |\partial_t f|^2 + V_{\varepsilon,\alpha}(t) f^2 - \frac{1}{2b}\left(2f^2 - f^4\right) \right\}$$

  where $V_{\varepsilon,\alpha}$ is approximately a shifted harmonic potential:

$$V_{\varepsilon,\alpha}(t) = \frac{(t+\alpha-\frac{1}{2}\varepsilon k t^2)^2}{(1-\varepsilon k t)^2} = (t+\alpha)^2 + \mathcal{O}(\varepsilon|\log\varepsilon|).$$

- For $1 < b < \Theta_0^{-1}$, $\exists\ \alpha(k) < 0$ and f_k minimizing $E^{1D}_{k,\alpha}$, i.e., such that

$$E_\star(k) = \inf_{\alpha\in\mathbb{R}} \inf_{f\in H^1} \mathcal{E}^{1D}_{k,\alpha}[f] = \mathcal{E}^{1D}_{k,\alpha(k)}[f_k].$$

Refined Energy Asymptotics

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Theorem (Energy asymptotics [MC, Rougerie '15])

Let Ω *be as in* (hp) *with boundary curvature* $\mathfrak{K}(\sigma)$. *Then, for any fixed* $1 < b < \Theta_0^{-1}$, *as* $\varepsilon \to 0$,

$$E_\varepsilon^{\mathrm{GL}} = \frac{1}{\varepsilon} \int_0^{|\partial\Omega|} \mathrm{d}\sigma \, E_\star(\mathfrak{K}(\sigma)) + \mathcal{O}(\varepsilon |\log \varepsilon|^\infty).$$

Remark

Expanding further $E_\star(\mathfrak{K}(\sigma))$, one gets [MC, Rougerie '16]

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| \, E_0^{\mathrm{1D}}}{\varepsilon} - E_{\mathrm{corr}} \int_0^{|\partial\Omega|} \mathrm{d}\sigma \mathfrak{K}(\sigma) + o(1) = \frac{|\partial\Omega| \, E_0^{\mathrm{1D}}}{\varepsilon} - 2\pi E_{\mathrm{corr}} + o(1),$$

$$E_{\mathrm{corr}} = \int_0^\infty \mathrm{d}t \, t \left\{ \left(f_0'\right)^2 + \left(-\alpha_0(t+\alpha_0) - \tfrac{1}{b} + \tfrac{1}{2b} f_0^2\right) f_0^2 \right\}.$$

$$\mathcal{E}_{\varepsilon}^{\mathrm{GL}}[\Psi, \mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r} \, \left\{ \left| \left(\nabla + i \tfrac{\mathbf{A}}{\varepsilon^2} \right) \Psi \right|^2 - \tfrac{1}{2b\varepsilon^2} \left(2|\Psi|^2 - |\Psi|^4 \right) + \tfrac{1}{\varepsilon^4} \left| \mathrm{curl} \mathbf{A} - 1 \right|^2 \right\}$$

Sketch of the Proof.

- Decompose the boundary layer $\mathcal{A}_{\varepsilon}$ into small cells, in such a way that one can approximate $\mathfrak{K}(\sigma) \simeq k$ const. inside each one;
- Prove the upper bound by evaluating $\mathcal{F}_{\varepsilon}$ on a trial state obtained by gluing k-dependent states $f_k(t) e^{-i\alpha_k s}$.
- Adapt the arguments leading to the lower bound for $k = 0$: construct a k-dependent potential function and prove that the corresponding k-dependent cost function is pointwise positive;
- Prove that $E_{\star}(k)$ and f_k are regular functions of k (at least C^1) to glue together all the contributions.

□

Proof of Pan's Conjecture

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Theorem (Density asymptotics [MC, Rougerie '15])

Let Ω be as in (hp). Then, for any fixed $1 < b < \Theta_0^{-1}$, as $\varepsilon \to 0$,

$$\left\| \left| \Psi^{\mathrm{GL}} \right| - f_0\left(0\right) \right\|_{L^\infty(\partial\Omega)} = \mathcal{O}(\varepsilon^{1/4} |\log \varepsilon|).$$

Remark

Similar estimate in a larger domain $\mathcal{A}_{\mathrm{bl}} \subset \{\mathrm{dist}(\mathbf{r}, \partial\Omega) \leqslant C\varepsilon\sqrt{|\log\varepsilon|}\}$:

$$\left\| \left| \Psi^{\mathrm{GL}} \right| (\,\cdot\,) - f_0\left(\mathrm{dist}(\,\cdot\,;\partial\Omega)/\varepsilon\right) \right\|_{L^\infty(\mathcal{A}_{\mathrm{bl}})} = o(1).$$

Corollary

Under the same assumptions, the degree of Ψ^{GL} along $\partial\Omega$ is well defined:

$$\deg\left(\Psi^{\mathrm{GL}}, \partial\Omega\right) = \frac{|\Omega|}{\varepsilon^2} - \frac{\alpha_0}{\varepsilon} + \mathcal{O}(\varepsilon^{-3/4}|\log\varepsilon|^\infty).$$

Curvature Corrections

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- To leading order $|\Psi^{\mathrm{GL}}|(\mathbf{r}) \simeq f_0(\mathrm{dist}(\mathbf{r}, \partial\Omega)/\varepsilon)$ and superconductivity is uniformly distributed in the boundary layer. Any lower order effect of the curvature?
- Recalling that $E^{\mathrm{1D}}_{\star}(k) = E^{\mathrm{1D}}_0 + \varepsilon k E_{\mathrm{corr}} + \mathcal{O}(\varepsilon^{3/2} |\log \varepsilon|^{\infty})$.

Theorem (Curvature corrections [MC, Rougerie *LMP* '16])

Let Ω be as in (hp). Then, for any fixed $1 < b < \Theta_0^{-1}$, as $\varepsilon \to 0$, and for any "rectangular" set D,

$$\frac{1}{2b} \int_D \mathrm{d}\mathbf{r} \, |\Psi^{\mathrm{GL}}|^4 = -\varepsilon E^{\mathrm{1D}}_0 |\partial\Omega \cap \partial D| + \varepsilon^2 E_{\mathrm{corr}} \int_{\partial D \cap \partial\Omega} \mathrm{d}\sigma \, \mathfrak{K}(\sigma) + o(\varepsilon^2).$$

Remark

The sign of E_{corr} determines whether superconductivity is attracted or repelled by points of large curvature.

Sign of the Correction

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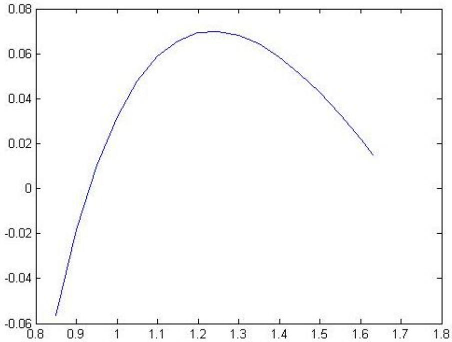
- Close to H_{c3}, superconductivity survives close to the points of maximal curvature.
- As $b \to \left(\Theta_0^{-1}\right)^-$, $f_0 \to 0$ and $E_{corr} \to 0$ but $E_{corr} > 0$.
- For $1 < b < \Theta_0^{-1}$, we investigated numerically the sign of E_{corr} and we found the same behavior [MC, Devanarayanan, Rougerie '18]:



Effect of Corners

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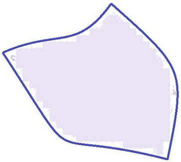
- So far we have considered only domains with smooth boundary. What happens if the boundary is not smooth but contains corners?
- The presence of corners might affect the boundary distribution of superconductivity.



- The third critical field H_{c3} can also be shifted because of corners.
- From now on we will assume that the boundary of Ω is a Lipschitz boundary with N corners with opening angles ϑ_j (hp).
- The normal $\boldsymbol{\nu}(\sigma)$ as well as tubular coordinates and the curvature $\mathfrak{K}(\sigma)$ are well-defined only a.e., with jumps at corners.

H_{c3} with Corners

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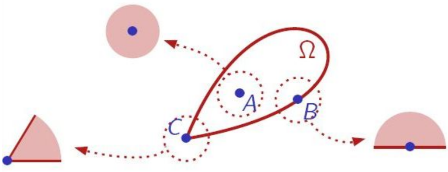

Courtesy of V. Bonnaillie-Noël.

- In presence of corners a new effective model appears: the magnetic Laplacian on a sector Γ_ϑ with opening angle $\vartheta \in [0, 2\pi)$, which is the angle at the vertex of the corner.
- We thus set

$$K := -\left(\nabla + \tfrac{1}{2} i \mathbf{r}^{\perp}\right)^2, \qquad \text{in } L^2(\Gamma_\vartheta),$$

with g.s.e. $\gamma(\vartheta)$.

Linear Model

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$$K = -\left(\nabla + \tfrac{1}{2} i \mathbf{r}^{\perp}\right)^2, \qquad \text{in } L^2(\Gamma_\vartheta)$$

- Unlike the magnetic Laplacian on the plane or on the half-plane, the behavior of the g.s.e. $\gamma(\vartheta)$ is not completly known.
- $\gamma(\vartheta) \to 0$ as $\vartheta \to 0$ [Bonnaillie-Noël, Dauge '06];
- $\gamma(\pi) = \Theta_0$;
- $\gamma(\vartheta) < \Theta_0$ if $\vartheta \leqslant \frac{\pi}{2} + \delta$ [Bonnaillie-Noël '05; Exner, Lotoreichik, Pérez-Obiol '17];

Conjecture ([Bonnaillie-Noël, Dauge '07])

Motivated by numerical computations, one expects that

- $\gamma(\vartheta)$ *is increasing in* ϑ;
- $\gamma(\vartheta) < \Theta_0$ *for* $\vartheta < \pi$;
- $\gamma(\vartheta) = \Theta_0$ *for* $\vartheta \geqslant \pi$.

A New Critical Field H_{corner}?

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Theorem (H_{c3} with corners [Bonnaillie-Noël, Fournais '07])

Let Ω *be as in* (hp). *Then, assuming that* $\gamma(\vartheta_j) < \Theta_0$ *for some* j,

$$H_{c3} = \frac{1}{\gamma_{\star}\varepsilon^2} + \mathcal{O}(1)$$

with $\gamma_{\star} = \min_j \gamma(\vartheta_j)$.

- H_{c3} is larger in presence of corners.
- Before disappearing, superconductivity gets concentrated near the corner with smallest opening angle and Ψ^{GL} decays exponentially in the distance from that corner.
- What happens to surface superconductivity? is there another critical field $H_{c2} < H_{\text{corner}} < H_{c3}$ marking the transition from boundary to corner concentration?

Surface Superconductivity

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Theorem (GL asymptotics [MC, Giacomelli '17])

Let Ω be as in (hp). Then, for any $1 < b < \Theta_0^{-1}$, as $\varepsilon \to 0$,

$$E_{\varepsilon}^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} + \mathcal{O}(|\log \varepsilon|^2).$$

Furthermore, the density $\left|\Psi^{\mathrm{GL}}\right|^2$ is L^2-close to f_0^2.

Remark

Based on the above result, one expects that H_{c2} is unaffected but, if the conjecture on the linear model is correct, one also expects that, in presence of at least one acute angle, a new critical field emerges:

$$H_{\mathrm{corner}} = \frac{1}{\Theta_0 \varepsilon^2} + \mathcal{O}(1).$$

Effect of Corners

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- The curvature $\mathfrak{K}(\sigma)$ for a Lipschitz boundary is still bounded and integrable and therefore we might expect the same energy asymptotics up to order $o(1)$:

$$E_{\varepsilon}^{\mathrm{GL}} \stackrel{?}{=} \frac{|\partial\Omega|\, E_0^{\mathrm{1D}}}{\varepsilon} - E_{\mathrm{corr}} \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \mathfrak{K}(\sigma) + o(1)$$

Theorem (GL refined asympt. [MC, Giacomelli '18])

Let Ω be as in (hp). Then, for any $1 < b < \Theta_0^{-1}$, as $\varepsilon \to 0$,

$$E_{\varepsilon}^{\mathrm{GL}} = \frac{|\partial\Omega|\, E_0^{\mathrm{1D}}}{\varepsilon} - E_{\mathrm{corr}} \int_0^{|\partial\Omega|} \mathrm{d}\sigma\, \mathfrak{K}(\sigma) + \sum_j E_{\mathrm{corner}}(\vartheta_j) + o(1).$$

Corner Energy

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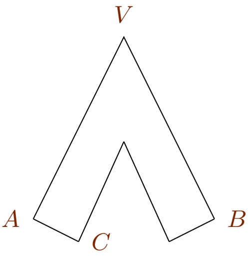
$$\mathcal{E}_1^{\mathrm{GL}}[\Psi, \mathbf{A}; D] = \int_D \mathrm{d}\mathbf{r} \left\{ |\nabla_{\mathbf{A}} \Psi|^2 - \frac{1}{2b} \left(2|\Psi|^2 - |\Psi|^4 \right) + |\mathrm{curl}\mathbf{A} - 1|^2 \right\}$$

- The corner energy is defined *implicitly* as

$$E_{\mathrm{corner}}(\vartheta) := \lim_{\ell \to +\infty} \lim_{L \to +\infty} \left(E_{\ell,L,\vartheta}^{\mathrm{GL}} - 2L E_0^{\mathrm{1D}} \right).$$

- $E_{\ell,L,\beta}^{\mathrm{GL}} = \inf_{\psi \in \mathscr{D}} \mathcal{E}_1^{\mathrm{GL}}[\psi, \frac{1}{2}\mathbf{r}^{\perp}; \Gamma_\vartheta(\ell, L)]$ and $2LE_0^{\mathrm{1D}}$ is the expected surface energy.
- $\Gamma_\vartheta(\ell, L)$ is a wedge-like domain of opening angle $\widehat{AVB} = \vartheta$ and with $\overline{AC} = \ell$, $\overline{AV} = L$.
- The magnetic potential is fixed to $\frac{1}{2}\mathbf{r}^{\perp}$.
- Ψ satisfies Dirichlet-type boundary conditions along $\overline{AC}$ to ensure that no surface energy is generated there.



Corner Energy

Conjecture (Corner energy [MC, Giacomelli '18])

For any $\vartheta \in [0, 2\pi)$, $E_{\mathrm{corner}}(\vartheta) = (\vartheta - \pi) E_{\mathrm{corr}}$.

Remark

For angles close to π, we can show [MC, Giacomelli '20] that

$$E_{\mathrm{corner}}(\pi - \epsilon) \underset{\epsilon \to 0}{=} -\epsilon E_{\mathrm{corr}} + o_{\epsilon}(1).$$

- Gauss-Bonnet theorem yields $\displaystyle\int_{\partial\Omega_{\mathrm{smooth}}} \mathrm{d}\sigma \, \mathfrak{K}(\sigma) + \sum (\pi - \vartheta_j) = 2\pi$, i.e., we can formally replace $\mathfrak{K}(\sigma) \longrightarrow \mathfrak{K}(\sigma) + \sum (\pi - \vartheta_j) \delta_{\sigma_j}$.
- The correction to the energy becomes

$$-E_{\mathrm{corr}} \int_0^{|\partial\Omega|} \mathrm{d}\sigma \, \mathfrak{K}(\sigma) \longrightarrow -E_{\mathrm{corr}} \left(\int_0^{|\partial\Omega|} \mathrm{d}\sigma \, \mathfrak{K}(\sigma) + \sum (\pi - \vartheta_j) \right) = -2\pi E_{\mathrm{corr}}$$

Conclusions

- As $\kappa \to \infty$, one can identify another critical field $H_{c2} = \kappa^2$, marking the transition from bulk to surface superconductivity;
- The energy and density can be approximated in terms of a one-dimensional effective model in the normal distance from $\partial\Omega$;
- To leading order energy and density are independent of the curvature, but the next-to-leading order expansion contains curvature-dependent terms;
- Superconductivity is defect-free in the surface layer (Pan's conjecture);
- In presence of corners with acute angles, $H_{c3} = \gamma_\star^{-1}\varepsilon^{-2}$ gets larger and a new critical field H_{corner} appears for the transition from surface to corner superconductivity;
- Corners affect the next-to-leading order term in the energy expansion, via a new model problem in an infinite sector.