Ginzburg-Landau Theory of Superconductivity

[lecture 2 – part I]

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DIPARTIMENTO DI MATEMATICA

Mathematical Methods in Field Theory and Quantum Mechanics

GSSI & SISSA

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Outline

1. Recap:
   - Minimization of the GL functional.
   - First and third critical fields.
   - Phenomenology.

- Surface superconductivity [Co]:
  2. Agmon estimates and decay in the bulk [FH]: second critical field H_{c2};
  3. Restriction of a boundary layer and replacement of the magnetic potential [FH];
  4. Energy & density asymptotics between H_{c2} and H_{c3} [CR1];

Main References

- [Co] M. C., *arXiv:2003.00521*, *Boll. Unione Mat. Italiana* online (2020).
- [CR1] M. C., N. Rougerie, *Commun. Math. Phys.* **332** (2014), *arXiv:1309.2268*.
- [FH] S. Fournais, B. Helffer, *Spectral Methods in Surface Superconductivity*, Progr. Nonlinear Differential Equations Appl. **77**, 2010.

GL Minimization

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- Minimization over $(\Psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega; \mathbb{C}^2)$ of

$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi, \mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r} \, \left\{ \left| (\nabla + i\mathbf{A}) \Psi \right|^2 - \kappa^2 |\Psi|^2 + \tfrac{1}{2}\kappa^2 |\Psi|^4 + |\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}|^2 \right\}$$

- Gauge invariance: $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi, \mathbf{A}] = \mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi e^{-i\phi}, \mathbf{A} + \nabla\phi]$.
- Euler-Lagrange eqs. $(\star)$ $\begin{cases} -\left(\nabla + i\mathbf{A}\right)^2 \Psi = \kappa^2 (1 - |\Psi|^2)\Psi, & \text{in } \Omega, \\ \nabla^\perp \left(\mathrm{curl}\mathbf{A}\right) = \mathbf{j}_{\mathbf{A}}[\Psi], & \text{in } \Omega, \\ \boldsymbol{\nu} \cdot \left(\nabla + i\mathbf{A}\right) \Psi = 0, & \text{on } \partial\Omega; \\ \mathrm{curl}\mathbf{A} = h_{\mathrm{ex}}, & \text{on } \partial\Omega; \end{cases}$
- There exists at least one ground state $(\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}, \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}})$ such that
  - any $(\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}, \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}})$ is smooth, solves $(\star)$ and $\left\| \Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \right\|_\infty \leqslant 1$;
  - $\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ is in the Coulomb gauge, i.e., $\begin{cases} \nabla \cdot \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} = 0, & \text{in } \Omega, \\ \boldsymbol{\nu} \cdot \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} = 0, & \text{on } \partial\Omega. \end{cases}$ and

  therefore $\left\| \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \right\|_{H^1(\Omega)} \leqslant C \left\| \mathrm{curl}\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \right\|_{L^2(\Omega)}$.

Critical Fields

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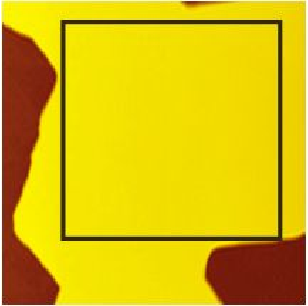
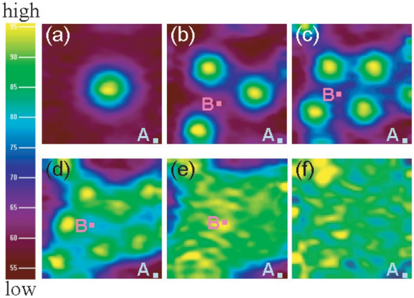
- As $\kappa \to \infty$, one can identify critical values (critical fields) of the applied field h_{ex}, marking the transition from one superconducting phase to another;
- (Meissner phase) below $H_{\mathrm{c1}}(\kappa) = \frac{1}{2|\xi_\star|} \log \kappa$, $\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ is defect free and $\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \simeq h_{\mathrm{ex}} \mathbf{A}_0$;
- (mixed phase) above $H_{\mathrm{c1}}(\kappa)$ vortices of $\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ start to appear and $\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ is non-zero there;
- (normal phase) above $H_{\mathrm{c3}}(\kappa) = \Theta_0^{-1} \kappa^2$, the normal state $(0, h_{\mathrm{ex}} \mathbf{F})$ is the unique ground state of $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$;
- the mixed phase is in fact much richer and one can distinguish between a bulk regime (superconductivity surviving in the interior of Ω) and a surface regime (superconductivity only at $\partial\Omega$)...

Phenomenology

- Before being totally *lost*, superconductivity survives at the boundary (surface superconductivity) as predicted by Saint-James, de Gennes in 1963 and observed by Strongin *et al* in 1964.



Pb island of superconductor at 4.32 K [Ning *et al* '09].



Vortices and surface superconductivity on a Pb island [Ning *et al* '09].

Agmon Estimates

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Proposition (Agmon estimates [Helffer, Morame '01])

Let Ω *be as in* (hp). *If* $h_{\mathrm{ex}} \sim \kappa^2$, *as* $\kappa \to +\infty$, *with* $b := \lim \frac{h_{\mathrm{ex}}}{\kappa^2} > 1$, $\exists c_b > 0$ *and* C *finite, such that, for any* $(\Psi, \mathbf{A})$ *solving* $(\star)$,

$$\int_\Omega \mathrm{d}\mathbf{r}\, e^{c_b\, \kappa\, \mathrm{dist}(\mathbf{r}, \partial\Omega)} \left\{ |\Psi|^2 + \tfrac{1}{\kappa^2} \left| \nabla_{\mathbf{A}} \Psi \right|^2 \right\} \leqslant C \int_{\{\mathrm{dist}(\mathbf{r},\partial\Omega) \leqslant \frac{1}{\kappa}\}} \mathrm{d}\mathbf{r}\, |\Psi|^2.$$

Remark

R.h.s. $= \mathcal{O}(\frac{1}{\kappa})$, which suggests that $\Psi = \mathcal{O}(\kappa^{-\infty})$ for $\mathrm{dist}(\mathbf{r}, \partial\Omega) \gg \frac{1}{\kappa}$.

Remark

The condition $b > 1$ is optimal, i.e.,

- if $b < 1$, there is no decay;
- if $b = 1$, the decay is polynomial [Fournais, Kachmar '11].

Elliptic Estimates

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Lemma

Let Ω *be as in* (hp). *For any* $(\Psi, \mathbf{A})$ *solving* ($\star$), *if* $h_{\mathrm{ex}} \sim \kappa^2$, $\kappa \to +\infty$,

$$\|\nabla_{\mathbf{A}}\Psi\|_{L^\infty(\Omega)} \leqslant C\kappa, \qquad \|\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}\|_{C^1(\overline{\Omega})} \leqslant C\kappa.$$

Remark

Since $\mathrm{curl}\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \simeq h_{\mathrm{ex}}$, we deduce that $\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \simeq h_{\mathrm{ex}}\mathbf{F}$ at least in $H^1(\Omega)$.

Idea of the Proof.

The first eq. in ($\star$) $-\nabla^2_{\mathbf{A}}\Psi = \kappa^2(1-|\Psi|^2)\Psi$ yields $\left\|\nabla^2_{\mathbf{A}}\Psi\right\|_2 \leqslant C\kappa^2$, which allows to bound the H^2-norm of Ψ and get the result by bootstrap. By the second eq. in ($\star$) $\nabla^{\perp}(\mathrm{curl}\mathbf{A}) = \mathbf{j}_{\mathbf{A}}[\Psi]$ and the above inequality,

$$\|\nabla(\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}})\|_\infty \leqslant C\,\|\mathbf{j}_{\mathbf{A}}[\Psi]\|_\infty \leqslant C\kappa,$$

The b.c. $\mathrm{curl}\mathbf{A} = h_{\mathrm{ex}}$ on $\partial\Omega$ implies $\|\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}\|_\infty \leqslant C\kappa$. □

Proof of Agmon Estimates.

Let $\chi \in C_0^\infty(\Omega)$, using that $\|\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}\|_{C^1(\overline{\Omega})} \leqslant C\kappa$ and

$$-\nabla_{\mathbf{A}}^2 \Psi = \kappa^2 (1 - |\Psi|^2)\Psi, \qquad \int_\Omega \mathrm{d}\mathbf{r}\ |\nabla_{\mathbf{A}} (\chi\Psi)|^2 \geqslant \int_\Omega \mathrm{d}\mathbf{r}\, \mathrm{curl}\mathbf{A}\ |\chi\Psi|^2,$$

we get

$$h_{\mathrm{ex}} \left(1 + \mathcal{O}(\tfrac{1}{\kappa})\right) \int_\Omega \mathrm{d}\mathbf{r}\ |\chi\Psi|^2 \leqslant \int_\Omega \mathrm{d}\mathbf{r}\ \left\{ |\nabla\chi|^2 |\Psi|^2 + \kappa^2 |\chi\Psi|^2 \right\}.$$

It just remains to pick a suitable χ: setting $t(\mathbf{r}) := \mathrm{dist}(\mathbf{r}, \partial\Omega)$,

$$\chi(\mathbf{r}) = e^{c_b \kappa t(\mathbf{r})} f(\kappa t(\mathbf{r})), \qquad f(x) = \begin{cases} 1, & \text{if } x \in [1, +\infty), \\ 0, & \text{if } x \in [-\infty, \frac{1}{2}], \end{cases}$$

for some $f \in C^\infty(\mathbb{R})$, so that

$$|\nabla\chi|^2 \leqslant (1+\epsilon) c_b^2 \kappa^2 \chi^2 + \tfrac{1}{\epsilon} \kappa^2 e^{c_b \kappa t(\mathbf{r})} |f'(\kappa t(\mathbf{r}))|^2.$$

□

Second Critical Field

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Definition (H_{c2})

The second critical field is defined to be $H_{c2}(\kappa) := \kappa^2$.

Change of units between H_{c2} and H_{c3}

$$\mathcal{E}_{\varepsilon}^{\mathrm{GL}}[\Psi, \mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \Psi \right|^2 - \frac{1}{2b\varepsilon^2} \left(2|\Psi|^2 - |\Psi|^4 \right) + \frac{1}{\varepsilon^4} \left| \mathrm{curl}\mathbf{A} - 1 \right|^2 \right\}$$

- $\mathbf{A}$ measured in units h_{ex}, i.e., $\mathbf{A} \to h_{\mathrm{ex}} \mathbf{A}$.
- We are interested in the regime $H_{c2} < h_{\mathrm{ex}} < H_{c3}$, i.e., $h_{\mathrm{ex}} \sim \kappa^2$.
- Change of units to $(\kappa, h_{\mathrm{ex}}) \to (\varepsilon, b)$ with

$$\varepsilon = \frac{1}{\sqrt{h_{\mathrm{ex}}}} \sim \frac{1}{\kappa} \ll 1, \qquad 1 < b < \Theta_0^{-1}.$$

- We denote by $(\Psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ and $E_{\varepsilon}^{\mathrm{GL}}$ any g.s. and the g.s. energy of the functional $\mathcal{E}_{\varepsilon}^{\mathrm{GL}}$, respectively.

Surface Behavior

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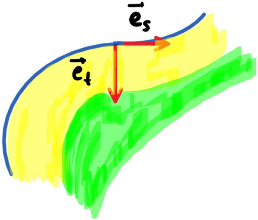
- By the Agmon estimates, Ψ^{GL} is concentrated in the region $\mathcal{A}_\varepsilon := \{\mathrm{dist}(\mathbf{r}, \partial\Omega) \leqslant c_0 \varepsilon |\log \varepsilon|\}$ and exponentially small in ε elsewhere.
- For ε small, $\exists$ diffeomorphism $\mathscr{F} : \mathcal{A}_\varepsilon \to [0, \frac{|\partial\Omega|}{\varepsilon}) \times [0, c_0 |\log \varepsilon|]$, such that $\mathbf{r}(s,t) =: \boldsymbol{\gamma}'(s) + \varepsilon t \boldsymbol{\nu}(s), \quad \varepsilon t := \mathrm{dist}\left\{\mathbf{r}, \partial\Omega\right\}$, where $\boldsymbol{\gamma} : [0, |\partial\Omega|/\varepsilon) \to \partial\Omega$.



- Let $\mathcal{E}_\varepsilon^{\mathrm{GL}}[\Psi, \mathbf{A}; \mathcal{D}]$ be the restriction of $\mathcal{E}_\varepsilon^{\mathrm{GL}}$ to a subdomain $\mathcal{D} \subset \Omega$.

Proposition (Domain restriction)

Let Ω be as in (hp). Then, if $b > 1$, as $\varepsilon \to 0$,

$$E_\varepsilon^{\mathrm{GL}} = \mathcal{E}_\varepsilon^{\mathrm{GL}}[\Psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}}; \mathcal{A}_\varepsilon] + \mathcal{O}(\varepsilon^\infty).$$

Magnetic Potential

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$$\mathcal{E}^{\mathrm{GL}}_{\varepsilon}[\Psi, \mathbf{A}; \mathcal{A}_\varepsilon] = \int_{\mathscr{F}(\mathcal{A}_\varepsilon)} \mathrm{d}\mathbf{r} \left\{ \left| \nabla_{\mathbf{A}/\varepsilon^2} \Psi \right|^2 - \frac{1}{2b\varepsilon^2} \left(2|\Psi|^2 - |\Psi|^4 \right) + \frac{1}{\varepsilon^4} \left| \mathrm{curl} \mathbf{A} - 1 \right|^2 \right\}$$

Proposition (Replacement of the magnetic potential)

Let Ω *be as in* (hp). *Then, if* $b > 1$, *as* $\varepsilon \to 0$, $\exists\ \phi_\varepsilon(s,t)$ *smooth, such that, setting* $\psi(s,t) := e^{-i\phi_\varepsilon(s,t)} \Psi^{\mathrm{GL}}(\mathbf{r}(s,t))$, *we get*

$$E^{\mathrm{GL}}_{\varepsilon} = \mathcal{F}_\varepsilon[\psi] + \mathcal{O}(1),$$

$$\mathcal{F}_\varepsilon[\psi] = \int_{\mathscr{F}(\mathcal{A}_\varepsilon)} \mathrm{d}s\mathrm{d}t \left\{ \left| \left(\nabla - it\mathbf{e}_s \right) \psi \right|^2 - \tfrac{1}{2b} \left(2|\psi|^2 - |\psi|^4 \right) \right\}.$$

Idea of the Proof.

$$\phi_\varepsilon(s,t) := -\frac{1}{\varepsilon} \left[\int_0^t \mathrm{d}\eta\, \mathbf{A}^{\mathrm{GL}}(\mathbf{r}(s,\eta)) \cdot \boldsymbol{\nu}(s) + \int_0^s \mathrm{d}\xi\, \mathbf{A}^{\mathrm{GL}}(\mathbf{r}(\xi,0)) \cdot \boldsymbol{\gamma}'(\xi) \right] + \mathcal{O}(\varepsilon)s$$

Drop the curvature dependent terms, e.g., in $\mathrm{d}\mathbf{r} \to \varepsilon^2 \mathrm{d}s\mathrm{d}t\,(1 - \varepsilon k(s) t)$. □

1D Effective Model

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$$\mathcal{F}_\varepsilon[\psi] = \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s \int_0^{c_0|\log\varepsilon|} \mathrm{d}t \left\{ \left| (\nabla - it\mathbf{e}_s)\,\psi \right|^2 - \tfrac{1}{2b}\left(2|\psi|^2 - |\psi|^4\right) \right\}.$$

Heuristics

We expect $|\psi|$ to be approx. constant in the s direction, while the phase of ψ is independent of t: for some $\alpha \in \mathbb{R}$ and $f : \mathbb{R}^+ \to \mathbb{R}$,

$$\psi(s,t) \simeq f(t) e^{-i\alpha s} \Longrightarrow \mathcal{F}_\varepsilon[\psi] \simeq \frac{|\partial\Omega|}{\varepsilon} \mathcal{E}^{\mathrm{1D}}_{0,\alpha}[f]$$

(current $\propto \frac{\alpha}{\varepsilon}$ along $\partial\Omega$, to compensate the magnetic field).

1D effective functional

$$\mathcal{E}^{\mathrm{1D}}_{0,\alpha}[f] := \int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \tfrac{1}{2b}\left(2f^2 - f^4\right) \right\}.$$

1D Effective Energy

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$$\mathcal{E}^{1D}_{0,\alpha}[f] = \int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \tfrac{1}{2b}\left(2f^2 - f^4\right) \right\}$$

Minimization of $\mathcal{E}^{1D}_{0,\alpha}$

- $\exists!$ minimizer $f_{0,\alpha} \geqslant 0$ with energy $E^{1D}_{0,\alpha}$.
- $f_{0,\alpha}$ is non-trivial iff $b^{-1} > \mu_0(\alpha)$, the g.s.e. of $H_\alpha = -\partial_t^2 + (t+\alpha)^2$ in $L^2(\mathbb{R}^+, \mathrm{d}t)$ with Neumann b.c. (recall that $\Theta_0 = \min_{\alpha \in \mathbb{R}} \mu_0(\alpha)$).

Optimal phase and density

If $1 \leqslant b < \Theta_0^{-1}$, $\exists$ $\alpha_0 < 0$ and f_0 minimizing $\mathcal{E}^{1D}_{0,\alpha}$, i.e., so that

$$E^{1D}_0 = \inf_{\alpha \in \mathbb{R}} \inf_{f \in H^1(\mathbb{R})} \mathcal{E}^{1D}_{0,\alpha}[f] = \mathcal{E}^{1D}_{0,\alpha_0}[f_0], \qquad \int_{\mathbb{R}^+} \mathrm{d}t\,(t+\alpha_0)\, f_0^2(t) = 0.$$

Energy and Density Asymptotics

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$$\mathcal{E}_\varepsilon^{\mathrm{GL}}[\Psi,\mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \tfrac{\mathbf{A}}{\varepsilon^2}\right) \Psi \right|^2 - \tfrac{1}{2b\varepsilon^2} \left(2|\Psi|^2 - |\Psi|^4\right) + \tfrac{1}{\varepsilon^4} \left| \mathrm{curl}\mathbf{A} - 1 \right|^2 \right\}$$

Theorem (GL asymptotics [MC, Rougerie '14])

Let Ω *be as in* (hp)*. Then, for any fixed* $1 \leqslant b < \Theta_0^{-1}$*, as* $\varepsilon \to 0$,

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{1\mathrm{D}}}{\varepsilon} + \mathcal{O}(1), \qquad \left\| |\Psi^{\mathrm{GL}}|^2(\,\cdot\,) - f_0^2\left(\tfrac{\mathrm{dist}(\,\cdot\,,\partial\Omega)}{\varepsilon}\right) \right\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon).$$

Remark

f_0 decays exponentially and $\left\| f_0^2\,(t) \right\|_{L^2(\mathcal{A}_\varepsilon)} \propto \varepsilon^{1/2}$.

Conjecture ([Pan '02])

The density $|\Psi^{\mathrm{GL}}|^2$ *is close to* $f_0^2(0)$ *in* $L^\infty(\partial\Omega)$.

Sketch of the Proof

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$$\mathcal{E}_{\varepsilon}^{\mathrm{GL}}[\Psi, \mathbf{A}] = \int_{\mathcal{A}_{\varepsilon}} \mathrm{d}\mathbf{r} \left\{ \left| \nabla_{\mathbf{A}/\varepsilon^2} \Psi \right|^2 - \frac{1}{2b\varepsilon^2} \left(2|\Psi|^2 - |\Psi|^4 \right) + \frac{1}{\varepsilon^4} \left| \mathrm{curl} \mathbf{A} - 1 \right|^2 \right\}$$

$$\mathcal{F}_{\varepsilon}[\psi] = \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s \int_0^{c_0 |\log\varepsilon|} \mathrm{d}t \left\{ \left| (\nabla - it\mathbf{e}_s)\, \psi \right|^2 - \tfrac{1}{2b} \left(2|\psi|^2 - |\psi|^4 \right) \right\}.$$

1. Restriction to the boundary layer & magnetic field replacement: already done with the energy reduction to $\mathcal{F}_{\varepsilon}[\psi]$.
2. Upper bound: evaluate $\mathcal{F}_{\varepsilon}$ on the trial state $\psi_{\mathrm{trial}}(s,t) \simeq f_0(t) e^{-i\alpha_0 s}$.

- Lower bound:
  3. Energy splitting.
  4. Use of the potential function.
  5. Positivity of the cost function.

6. Derivation of the estimate on the density $|\psi|^2$.

Energy Reduction & Upper Bound

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$$\mathcal{F}_\varepsilon[\psi] = \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s \int_0^{c_0|\log\varepsilon|} \mathrm{d}t \left\{ \left|(\nabla - it\mathbf{e}_s)\,\psi\right|^2 - \tfrac{1}{2b}\left(2|\psi|^2 - |\psi|^4\right) \right\}$$

❶ Energy Reduction

Agmon estimates & a suitable (local) gauge choice (with $\psi = e^{-i\phi_\varepsilon}\Psi^{\mathrm{GL}}$):

$$E_\varepsilon^{\mathrm{GL}} = \mathcal{F}_\varepsilon[\psi] + \mathcal{O}(1) \geqslant \inf_\psi \mathcal{E}_{\mathrm{hp}}[\psi] + \mathcal{O}(1),$$

$$\mathcal{E}_{\mathrm{hp}}[\psi] = \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}\sigma \mathrm{d}t \left\{ \left|(\nabla - it\mathbf{e}_\sigma)\,\psi\right|^2 - \frac{1}{b}|\psi|^2 + \frac{1}{2b}|\psi|^4 \right\}$$

❷ Upper Bound

Evaluate $\mathcal{F}_\varepsilon$ on $\psi_{\mathrm{trial}}(s,t) := f_0(t)e^{-i\widetilde{\alpha}_0 s}$, where $\widetilde{\alpha}_0 = \frac{2\pi\varepsilon}{|\partial\Omega|}\left\lfloor \frac{|\partial\Omega|}{2\pi\varepsilon} \right\rfloor$:

$$\mathcal{F}_\varepsilon[\psi] \leqslant \frac{|\partial\Omega|}{\varepsilon} E_0^{\mathrm{1D}} + \mathcal{O}(1).$$

Lower Bound (I)

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$$\mathcal{E}_{\mathrm{hp}}[\psi] = \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}s\mathrm{d}t \left\{ \left| \left(\nabla - it\mathbf{e}_\sigma \right) \psi \right|^2 - \frac{1}{b}|\psi|^2 + \frac{1}{2b}|\psi|^4 \right\}$$

❸ Energy splitting

- If $1 \leqslant b < \Theta_0^{-1}$, one can set $\psi(s,t) =: f_0(t) e^{-i\alpha_0 s} v(s,t)$.
- Using the variational equation of f_0 and its boundary conditions

$$\mathcal{E}_{\mathrm{hp}}[\psi] = \frac{|\partial\Omega|}{\varepsilon} E_0^{1\mathrm{D}} + \mathcal{E}[v]$$

with $\mathbf{j}[v] = \Im \left(v \nabla v^* \right)$ the current and

$$\mathcal{E}[v] = \int \mathrm{d}s\mathrm{d}t \, f_0^2 \left\{ |\nabla v|^2 - 2(t+\alpha_0)\mathbf{e}_s \cdot \mathbf{j}[v] + \frac{1}{2b} f_0^2 \left(1 - |v|^2 \right)^2 \right\}$$

- It remain to bound $\mathcal{E}[v]$ and we will eventually show that $\mathcal{E}[v] \geqslant 0$.

Lower Bound (II)

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$$\mathcal{E}[v] = \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}s\mathrm{d}t\, f_0^2 \left\{ |\nabla v|^2 - 2(t+\alpha_0)\mathbf{e}_s \cdot \mathbf{j}[v] + \frac{1}{2b} f_0^2 \left(1 - |v|^2\right)^2 \right\}$$

❹ Use of the potential function

- The field $2(t+\alpha_0) f_0^2 \mathbf{e}_\sigma$ is divergence free so that one can find F such that $\nabla^\perp F = 2(t+\alpha_0) f_0^2 \mathbf{e}_\sigma$, e.g., the potential function

$$F_0(t) = 2 \int_0^t \mathrm{d}\eta\, (\eta + \alpha_0) f_0^2(\eta).$$

- $F_0(0) = F_0(+\infty) = 0$ (optimality of α_0), $F_0'(0) < 0 \Longrightarrow F_0 \leqslant 0$.
- Stokes formula yields (vorticity measure $\mu = \mathrm{curl}(\mathbf{j})$, $|\mu| \leqslant |\nabla v|^2$)

$$\mathcal{E}[v] = \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}s\mathrm{d}t\, \left\{ f_0^2 |\nabla v|^2 + F_0 \mu + \frac{1}{2b} f_0^4 \left(1 - |v|^2\right)^2 \right\}$$

$$\geqslant \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}s\mathrm{d}t\, \left\{ f_0^2 |\nabla v|^2 + F_0 |\mu| \right\} \geqslant \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}s\mathrm{d}t\, \left(f_0^2 + F_0 \right) |\nabla v|^2 .$$

Lower Bound (III)

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$$\mathcal{E}[v] \geqslant \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}s\mathrm{d}t \, \left(f_0^2 + F_0\right) |\nabla v|^2 .$$

❺ Positivity of the cost function

- We define the vortex cost function as
$$K_0(t) = f_0^2(t) + F_0(t).$$
- If $1 \leqslant b < \Theta_0^{-1}$, $K_0(t) \geqslant 0$, for any $t \in \mathbb{R}^+$, which allows to conclude that $\mathcal{E}[u] \geqslant 0$ and the lower bound is proven.
- Optimality condition + variational equation for f_0 imply that
$$K_0(t) = \left(1 - \tfrac{1}{b}\right) f_0^2(t) + (t + \alpha_0)^2 f_0^2(t) + \tfrac{1}{2b} f_0^4(t) - {f_0'}^2(t)$$
- $K_0(0) > 0$ and $K_0(+\infty) = 0 \Longrightarrow$ if $K < 0$ somewhere, $\exists t_0 > 0$ global minimum point of K_0 and $K_0'(t_0) = 0$.
- Since $K_0' = 2 f_0 f_0' + 2\,(t + \alpha_0)\, f_0^2$, $f_0'(t_0) = -(t_0 + \alpha_0) f_0(t_0)$ and
$$K_0(t_0) = \left(1 - \tfrac{1}{b}\right) f_0^2(t_0) + \tfrac{1}{2b} f_0^4(t_0) \; \geqslant 0$$

Density Asymptotics

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$$\mathcal{E}[v] = \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s \int_0^{c_0|\log\varepsilon|} \mathrm{d}t \;\; f_0^2 \left\{ |\nabla v|^2 - 2(t+\alpha_0)\mathbf{e}_\sigma \cdot \mathbf{j}[v] + \frac{1}{2b} f_0^2 \left(1-|v|^2\right)^2 \right\}$$

❺ Density asymptotics

- Combining the lower bound

$$\mathcal{F}_\varepsilon[\psi] \geqslant \frac{|\partial\Omega|}{\varepsilon} E_0^{\mathrm{1D}} + \mathcal{E}[v] + \mathcal{O}(\varepsilon^\infty)$$

$$\geqslant \frac{|\partial\Omega|}{\varepsilon} E_0^{\mathrm{1D}} + \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s \int_0^{c_0|\log\varepsilon|} \mathrm{d}t \;\; \frac{1}{2b} f_0^4 \left(1-|v|^2\right)^2 + \mathcal{O}(\varepsilon^\infty),$$

with the upper bound $\mathcal{F}_\varepsilon[\psi] \leqslant \frac{|\partial\Omega|}{\varepsilon} E_0^{\mathrm{1D}} + \mathcal{O}(1)$, we get

$$\int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s \int_0^{c_0|\log\varepsilon|} \mathrm{d}t \; f_0^4 \left(1-|v|^2\right)^2 = \mathcal{O}(1).$$

- Recalling that $|\Psi^{\mathrm{GL}}| = f_0|v|$, we get the result.

Consequences

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$$\mathcal{E}[v] \geqslant \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}s\mathrm{d}t \left\{ f_0^2 \left|\nabla v\right|^2 + F_0|\mu| \right\}$$

Vortex energy cost

The cost function $K_0(t)$ gives the energy cost of a vortex at distance εt from $\partial\Omega$:

- the energy gain is $F_0\mu$: for a vortex with degree $n \in \mathbb{N}$ at $\mathbf{r}_0$, $\mu \simeq 2\pi n \delta_{\mathbf{r}_0}$, so that one lower the energy of $-2\pi n|F_0(t_0)|$;
- to accomodate a vortex at $\mathbf{r}_0$, v must vanish there and therefore $|\nabla v|$ is increased (energy loss);
- the pointwise positivity of K_0 for $1 \leqslant b < \Theta_0^{-1}$ implies that a vortex is nowhere energetically convenient in the surface layer and therefore superconductivity is defect free there.