Ginzburg-Landau Theory of Superconductivity

[lecture 1 – part II]

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DIPARTIMENTO DI MATEMATICA

Mathematical Methods in Field Theory and Quantum Mechanics
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1. Introduction [Ti]:
   - Response of a superconductor to an applied magnetic field;
   - Extreme states.

- Critical magnetic fields [FH,SS]:
  2. Weak magnetic fields: Meissner state;
  3. First critical field and occurrence of vortices;
  4. Third critical field and transition to normal state.

Main References

- [FH] S. Fournais, B. Helffer, *Spectral Methods in Surface Superconductivity*, Progr. Nonlinear Differential Equations Appl. **77**, 2010.
- [SS] E. Sandier, S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Progr. Nonlinear Differential Equations Appl. **70**, 2007.
- [Ti] M. Tinkham, *Introduction to Superconductivity*, 2004.

Response to a Magnetic Field

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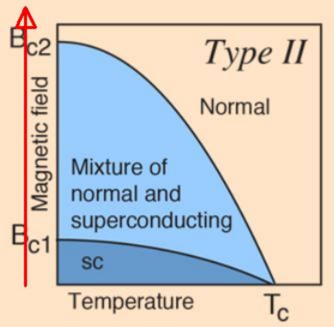
Weak magnetic fields are expelled from the sample (Meissner effect): superconductivity is preserved in the bulk and the magnetic field decays rapidly inside.

Strong magnetic fields can penetrate the sample and eventually destroy superconductivity. The penetration occurs first at isolated points and then expands in the whole bulk of the sample.



Response of a type-II superconductor [HyperPhysics].

Phenomenology (I)

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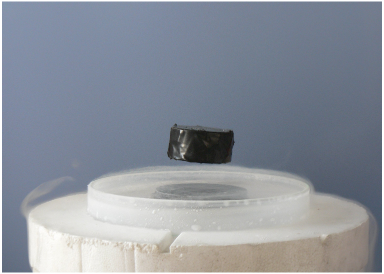
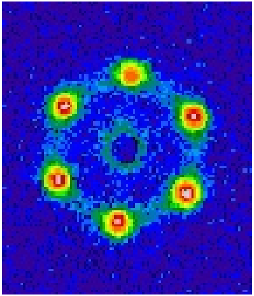
- Weak magnetic fields do not penetrate the superconducting sample (Meissner effect), i.e., the magnetic field is practically zero inside.
- Superconductivity is first lost at isolated defects, i.e., points where the magnetic field is non-zero and the order parameter has a non-trivial winding number (vortices).



Meissner effect
[Wikipedia].



Vortices in Nb *crystal*
[Ling *et al* '00].

Phenomenology (II)

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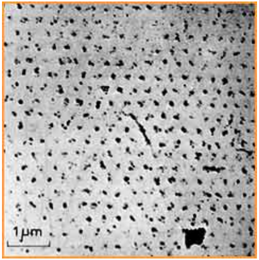
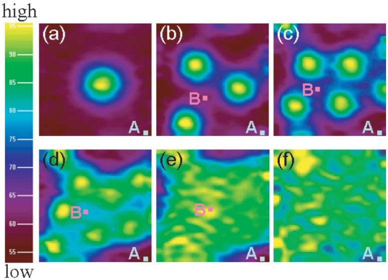
- The number of vortices increases with the magnetic field and eventually triangular lattices appear (predicted by Abrikosov in 1957 and observed by Essmann, Trauble in 1967).
- Before being totally lost, superconductivity survives at the boundary (surface superconductivity) (predicted by Saint-James, de Gennes in 1963 and observed by Strongin *et al* in 1964).



Vortices in Pb at 1.1 K
[Essmann, Trauble '67]



Surface superconductivity on Pb island at 4.32 K
[Ning et al '09]

Extreme States (I)

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$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r}\, \left\{ \left| (\nabla + i\mathbf{A})\, \Psi \right|^2 - \kappa^2 |\Psi|^2 + \tfrac{1}{2}\kappa^2 |\Psi|^4 + \left| \mathrm{curl}\mathbf{A} - h_{\mathrm{ex}} \right|^2 \right\}$$

Perfect Superconductor

The perfectly superconducting state is $(1, 0)$, i.e., all electrons arranged in Cooper pairs and no magnetic field inside (Meissner state), with energy

$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[1,0] = \left(-\tfrac{1}{2}\kappa^2 + h_{\mathrm{ex}}^2 \right) |\Omega|.$$

Normal Conductor

The normal state is $(0, \mathbf{A}_{\mathrm{ex}})$ with $\mathrm{curl}\mathbf{A}_{\mathrm{ex}} = h_{\mathrm{ex}}$, i.e., no electrons arranged in Cooper pairs and magnetic field h_{ex} inside, with energy

$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[0,\mathbf{A}_{\mathrm{ex}}] = 0.$$

Extreme States (II)

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$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] + \tfrac{1}{2}\kappa^2|\Omega| = \int_\Omega \mathrm{d}\mathbf{r}\, \left\{ \left|\nabla_{\mathbf{A}}\Psi\right|^2 + \tfrac{1}{2}\kappa^2\left(1-|\Psi|^2\right)^2 + \left|\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}\right|^2 \right\}$$

- If $h_{\mathrm{ex}} = 0$, the Meissner state $(1,0)$ is the unique (up to gauge transformations) ground state of $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$;
- If $\kappa = 0$, the normal state $(0,\mathbf{A}_{\mathrm{ex}})$ is the unique ground state;
- Whenever $h_{\mathrm{ex}} < \kappa/\sqrt{2}$, the Meissner state (energy $\left(-\frac{1}{2}\kappa^2 + h_{\mathrm{ex}}^2\right)|\Omega|$) beats the normal state (energy 0);
- Any state in order to be a ground state must have negative energy;
- For small h_{ex}, we should expect $|\Psi| \simeq 1$ and $\mathbf{A} \simeq 0$ *but*, if $h_{\mathrm{ex}} > 0$, $|\Psi| \not\equiv 1$: $(1,0)$ can not solve ($\star$)

$$\begin{cases} \nabla^{\perp}(\mathrm{curl}\mathbf{A}) = \mathbf{j}_{\mathbf{A}}[\Psi], & \text{in } \Omega; \\ \mathrm{curl}\mathbf{A} = h_{\mathrm{ex}}, & \text{on } \partial\Omega. \end{cases}$$

Meissner State (I)

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$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r}\, \left\{ \left| (\nabla + i\mathbf{A})\,\Psi \right|^2 - \kappa^2|\Psi|^2 + \tfrac{1}{2}\kappa^2|\Psi|^4 + \left|\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}\right|^2 \right\}$$

Proposition (Meissner magnetic potential)

Let Ω as in (hp). Then, $\exists!$ minimizer $\mathbf{A}_0$ w.r.t. $\mathbf{A} \in H^1_{\mathrm{div}}(\Omega;\mathbb{C}^2)$ of $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[1,\mathbf{A}]$ in the Coulomb gauge and it is given by

$$\mathbf{A}_0 = h_{\mathrm{ex}} \nabla^\perp h_0, \qquad \begin{cases} -\Delta h_0 + h_0 = 0, & \text{in } \Omega, \\ h_0 = 1, & \text{on } \partial\Omega. \end{cases}$$

Proof.

The EL eqs. for $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[1,\mathbf{A}] = \|\mathbf{A}\|^2_{L^2(\Omega)} + \|\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}\|^2_{L^2(\Omega)}$ read

$$\begin{cases} -\Delta \mathbf{A}_0 + \mathbf{A}_0 = 0, & \text{in } \Omega, \\ \mathrm{curl}\mathbf{A}_0 = h_{\mathrm{ex}}, & \text{on } \partial\Omega. \end{cases}$$

□

Meissner State (II)

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$$\begin{cases} -\Delta h_0 + h_0 = 0, & \text{in } \Omega, \\ h_0 = 1, & \text{on } \partial\Omega. \end{cases}$$

Exercise

Prove that $h_0 \in C^\infty(\overline{\Omega})$ and $0 \leqslant h_0 \leqslant 1$.

Proposition

Let Ω be as in (hp). Then, for any $\mathbf{r} \in \Omega$,

$$h_0(\mathbf{r}) \leqslant e^{-\frac{1}{2}\mathrm{dist}^2(\mathbf{r},\partial\Omega)}.$$

Idea of the Proof.

The function $W(\mathbf{r}) := e^{-\frac{1}{2}\mathrm{dist}^2(\mathbf{r},\partial\Omega)}$ is a supersolution for the problem solved by h_0: close to $\partial\Omega$, $-\Delta W + W = (2 - \mathrm{dist}(\mathbf{r}, \partial\Omega))\, W \geqslant 0$ and $W = 1$ on $\partial\Omega$. □

Vortices

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Definition (Vortices)

$\Psi : \Omega \to \mathbb{C}$ *has a vortex of winding number* $n \in \mathbb{Z}$ *at the point* $\mathbf{r}_0 \in \Omega$*, if*

$$\Psi(\mathbf{r}_0) = 0, \qquad \deg(\Psi, \partial\mathcal{B}_\varrho(\mathbf{r}_0)) := \frac{1}{2\pi} \int_{\partial\mathcal{B}_\varrho(\mathbf{r}_0)} \mathrm{d}s \, \frac{\Psi^*}{|\Psi|} \partial_\tau \left(\frac{\Psi}{|\Psi|} \right) = n.$$

i.e., $\frac{\Psi}{|\Psi|} = \left(\frac{z - z_0}{|z - z_0|} \right)^n$ *around* $\mathbf{r}_0$ *and in complex coordinates* $z \in \mathbb{C}$*.*

Example

$f(r) e^{in\vartheta}$ ($\mathbf{r} = (r, \vartheta) \in \mathbb{R}^+ \times [0, 2\pi)$) has a vortex of degree $n \in \mathbb{Z}$ at 0.

Heuristics

The penetration of the magnetic fields occurs first at the points where Ψ vanishes and a non-trivial winding number of Ψ there lowers the energy.

First Critical Field (I)

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Definition (H_{c1})

The first critical field is defined as

$$H_{c1}(\kappa) := \sup \left\{ h_{ex} > 0 \;\middle|\; \Psi^{GL}_{\kappa,h_{ex}} \text{ does not vanish in } \overline{\Omega} \right\}.$$

Proposition (Energy upper bounds)

Let Ω *as in* (hp) *and* $E_0 := \inf_{\mathbf{A}} \mathcal{F}^{GL}_{\kappa,h_{ex}}[1, \mathbf{A}] = \mathcal{F}^{GL}_{\kappa,h_{ex}}[1, \mathbf{A}_0]$, *then*

$$E^{GL}_{\kappa,h_{ex}} + \tfrac{1}{2}\kappa^2|\Omega| \leqslant \begin{cases} h_{ex}^2 E_0 = \mathcal{F}^{GL}_{\kappa,h_{ex}}[1, \mathbf{A}_0], & (\textit{no vortex}) \\ h_{ex}^2 E_0 + 2\pi \log\kappa - 4\pi|\xi_\star| h_{ex} + \mathcal{O}(1), & (1 \textit{ vortex}) \end{cases}$$

as $\kappa \to +\infty$ *and where* $\xi_\star := \min_{\mathbf{r}\in\Omega} h_0(\mathbf{r}) - 1$.

- A 1-vortex state beats a state without vortices, if $h_{ex} > \frac{1}{2|\xi_\star|} \log\kappa$.

$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] + \tfrac{1}{2}\kappa^2|\Omega| = \int_\Omega \mathrm{d}\mathbf{r}\,\left\{\left|\nabla_{\mathbf{A}}\Psi\right|^2 + \tfrac{1}{2}\kappa^2\left(1-|\Psi|^2\right)^2 + \left|\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}\right|^2\right\}$$

Proof.

Let $\mathbf{r}_0 \in \Omega$ be a minimum point for h_0, i.e., $\xi_\star = h_0(\mathbf{r}_0) - 1$, then set

$$\psi_{\mathrm{trial}}(\mathbf{r}) = \begin{cases} (z-z_0)/|z-z_0|, & \text{for } \mathbf{r} \in \mathcal{B}^c_{1/\kappa}(\mathbf{r}_0), \\ \kappa(z-z_0), & \text{otherwise.} \end{cases}$$

$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\psi_{\mathrm{trial}}, h_{\mathrm{ex}}\mathbf{A}_0] + \tfrac{1}{2}\kappa^2|\Omega| = \int_\Omega \mathrm{d}\mathbf{r}\,\left\{\left|\nabla_{h_{\mathrm{ex}}\mathbf{A}_0}\psi_{\mathrm{trial}}\right|^2 + h_{\mathrm{ex}}^2\left|h_0 - 1\right|^2\right\} + \mathcal{O}(1)$$

$$\leqslant \int_\Omega \mathrm{d}\mathbf{r}\,\left|\nabla\psi_{\mathrm{trial}}\right|^2 + 2h_{\mathrm{ex}}\int_{\Omega\setminus\mathcal{B}_{1/\kappa}(\mathbf{r}_0)} \mathrm{d}\mathbf{r}\,\mathbf{j}[\psi_{\mathrm{trial}}]\cdot\nabla^\perp h_0 + h_{\mathrm{ex}}^2 E_0 + \mathcal{O}(1)$$

$$\leqslant h_{\mathrm{ex}}^2 E_0 + 2\pi\int_{1/\kappa}^{R} \mathrm{d}r\,\frac{1}{r} + 2h_{\mathrm{ex}}\int_{\partial\left(\Omega\setminus\mathcal{B}_{1/\kappa}(\mathbf{r}_0)\right)} \mathrm{d}s\,h_0\boldsymbol{\tau}\cdot\nabla\phi_{\mathrm{trial}} + \mathcal{O}(1)$$

$$= h_{\mathrm{ex}}^2 E_0 + 2\pi\log\kappa + 4\pi h_{\mathrm{ex}}(1 - h_0(\mathbf{r}_0)) + \mathcal{O}(1).$$

□

First Critical Field (II)

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Theorem (H_{c1} [Sandier, Serfaty '03])

Let Ω *be as in* (hp). *Then, as* $\kappa \to +\infty$,

$$H_{c1} = \frac{1}{2|\xi_\star|} |\log \kappa|.$$

Moreover, if $h_{ex} \leqslant H_{c1} - c \log\log\kappa$, *then* $E^{GL}_{\kappa,h_{ex}} + \frac{1}{2}\kappa^2|\Omega| = h_{ex}^2 E_0 + o(1)$,

- $\Psi^{GL}_{\kappa,h_{ex}}$ *is vortex free and* $|\Psi^{GL}_{\kappa,h_{ex}}| \geqslant 1 + o(1)$ *in* $\overline{\Omega}$;
- $\mathbf{A}^{GL}_{\kappa,h_{ex}} = h_{ex}\mathbf{A}_0(1 + o(1))$ *(up to gauge transf.) in* $H^1(\Omega; \mathbb{C}^2)$;

while, if $h_{ex} \geqslant H_{c1} + c_\kappa$, $c_\kappa \gg 1$, *then*

- $\Psi^{GL}_{\kappa,h_{ex}}$ *has at least one vortex and* $\mathbf{A}^{GL}_{\kappa,h_{ex}} = h_{ex}\nabla^\perp h$ *with*

$$\begin{cases} -\Delta h + h = 2\pi \sum n_i \delta_{\mathbf{r}_i}, & \textit{in } \Omega, \\ h = 1, & \textit{on } \partial\Omega; \end{cases}$$

- *with* 1 *vortex,* $E^{GL}_{\kappa,h_{ex}} + \frac{1}{2}\kappa^2|\Omega| = h_{ex}^2 E_0 + 2\pi\log\kappa - 4\pi|\xi_\star|h_{ex} + \mathcal{O}(1)$.

Third Critical Field (I)

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$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r}\,\left\{ \left| (\nabla + i\mathbf{A})\,\Psi \right|^2 - \kappa^2|\Psi|^2 + \tfrac{1}{2}\kappa^2|\Psi|^4 + \left|\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}\right|^2 \right\}$$

- Close to the transition to the normal state $|\Psi| \equiv 0$, we expect that the nonlinear term can be dropped (linear model);
- The magnetic potential should be close to $h_{\mathrm{ex}}\mathbf{F}$, such that $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[0, h_{\mathrm{ex}}\mathbf{F}] = \inf_{\mathbf{A}} \mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[0,\mathbf{A}]$, i.e., the unique solution of

$$\begin{cases} \mathrm{curl}\mathbf{F} = 1, & \text{in } \Omega; \\ \nabla \cdot \mathbf{F} = 0, & \text{in } \Omega; \\ \boldsymbol{\nu} \cdot \mathbf{F} = 0, & \text{on } \partial\Omega. \end{cases}$$

Definition (H_{c3})

The third critical field is defined as

$$H_{\mathrm{c3}}(\kappa) := \inf \left\{ h_{\mathrm{ex}} > 0 \,\middle|\, (0, h_{\mathrm{ex}}\mathbf{F}) \text{ is a g.s. of } \mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \right\}.$$

Third Critical Field (II)

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$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] \simeq \int_\Omega \mathrm{d}\mathbf{r}\ \left\{ \left| (\nabla + i h_{\mathrm{ex}} \mathbf{F})\, \Psi \right|^2 - \kappa^2 |\Psi|^2 \right\}$$

Exercise

Prove the following inequality: for any $\Psi \in C_0^\infty(\Omega)$ and $\mathbf{A} \in H^1(\Omega;\mathbb{C}^2)$,

$$\int_\Omega \mathrm{d}\mathbf{r}\ \left| (\nabla + i\mathbf{A})\, \Psi \right|^2 \geqslant \left| \int_\Omega \mathrm{d}\mathbf{r}\, \mathrm{curl}\mathbf{A}\ |\Psi|^2 \right|.$$

[*HINT:* write $\mathrm{curl}\mathbf{A} = -i\left[\partial_x + i\mathbf{A}_x, \partial_y + i\mathbf{A}_y\right]$ and use Cauchy-Schwarz]

Heuristics

Applying the inequality, we get

$$E^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \gtrsim \left(h_{\mathrm{ex}} - \kappa^2\right) \|\Psi\|_2^2\,,$$

so that, if $h_{\mathrm{ex}} \gg \kappa^2$, the best choice is $\Psi \equiv 0$ (normal state).

Magnetic Schrödinger operators (I)

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$$H = -\left(\nabla + \tfrac{1}{2} i h \mathbf{r}^{\perp}\right)^2, \qquad \text{in } L^2(\mathbb{R}^2)$$

Proposition (GSE on $\mathbb{R}^2$)

For any $h > 0$, $\inf \sigma(H) = h$.

Proof.

Via the gauge transformation $\Psi \to e^{-\frac{1}{2} i x y} \Psi$, we can map H to $-\left(\nabla + i h(-y, 0)\right)^2$ and, by partial Fourier transform in x and rescaling $y \to h y$, we reduce to $h H_\alpha$ with

$$H_\alpha = -\partial_y^2 + (y + \alpha)^2, \qquad \text{on } L^2(\mathbb{R}) \text{ with } \alpha \in \mathbb{R}.$$

The ground state energy is 1 (independent of α) and the ground state is $\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(y+\alpha)^2}$, which yields the optimizer of the previous inequality. □

Magnetic Schrödinger operators (II)

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$$H_+ = -\left(\nabla + \tfrac{1}{2} i h \mathbf{r}^\perp\right)^2, \qquad \text{in } L^2(\mathbb{R}^2_+)$$

Proposition (GSE on $\mathbb{R}^2_+$)

For any $h > 0$, $\inf \sigma(H_+) = \Theta_0 h$, with $\Theta_0 \simeq 0.59$.

Sketch of the Proof.

Acting as before, we get the (Neumann realization of) operator

$$H_\alpha = -\partial_y^2 + (y + \alpha)^2, \qquad \text{on } L^2(\mathbb{R}^+) \text{ with } \alpha \in \mathbb{R}.$$

The ground state energy is $\inf \sigma(H_\alpha) =: \mu_0(\alpha)$ and depends on $\alpha \in \mathbb{R}$.

$$\Theta_0 := \min_{\alpha \in \mathbb{R}} \mu_0(\alpha) = \mu_0\left(-\sqrt{\Theta_0}\right).$$

□

Third Critical Field (III)

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$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r}\, \left\{ \left| (\nabla + i\mathbf{A})\,\Psi \right|^2 - \kappa^2 |\Psi|^2 + \tfrac{1}{2}\kappa^2 |\Psi|^4 + \left| \mathrm{curl}\mathbf{A} - h_{\mathrm{ex}} \right|^2 \right\}$$

Theorem (H_{c3} [Lu, Pan '00; Fournais, Helffer '06])

Let Ω be as in (hp). Then, as $\kappa \to +\infty$,

$$H_{c3} = \Theta_0^{-1}\kappa^2 + \mathcal{O}(\kappa).$$

Idea of the Proof.

- $\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ is close to $h_{\mathrm{ex}}\mathbf{F}$ in $L^4(\Omega)$ and the latter can be replaced with $\frac{1}{2}h_{\mathrm{ex}}\mathbf{r}^\perp + \mathcal{O}(|\mathbf{r}|^2)$ via change of gauge;
- Since $h_{\mathrm{ex}} \gg 1$ and the lenght scale of g.s. is h_{ex}^{-1}, we recover 2 model Schrödinger operators: H in the interior of Ω and H_+ close to $\partial\Omega$;
- H_+ yields the lowest energy: $\mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi, h_{\mathrm{ex}}\mathbf{F}] \gtrsim \left(\Theta_0 h_{\mathrm{ex}} - \kappa^2\right) \|\Psi\|_2^2$.

□

Conclusions

- As $\kappa \to \infty$, one can identify critical values (critical fields) of the applied field h_{ex}, marking the transition from one superconducting phase to another;
- (Meissner phase) below $H_{\mathrm{c1}}(\kappa) = \frac{1}{2|\xi_\star|} \log \kappa$, $\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ is defect free and $\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \simeq h_{\mathrm{ex}} \mathbf{A}_0$;
- (mixed phase) above $H_{\mathrm{c1}}(\kappa)$ vortices of $\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ start to appear and $\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ is non-zero there;
- (normal phase) above $H_{\mathrm{c3}}(\kappa) = \Theta_0^{-1} \kappa^2$, the normal state $(0, h_{\mathrm{ex}} \mathbf{F})$ is the unique ground state of $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$;
- the mixed phase is in fact much richer and one can distinguish between a bulk regime (superconductivity surviving in the interior of Ω) and a surface regime (superconductivity only at $\partial\Omega$)...