Ginzburg-Landau Theory of Superconductivity

[lecture 1 – part I]

Michele Correggi

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Outline

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1. Introduction [Ti]:
   - Superconductivity;
   - Response of a superconductor to an applied magnetic field.

- Ginzburg-Landau(GL) theory [FH,SS]:
  2. Gauge invariance;
  3. Minimization of the GL functional;
  4. Maximum principle.

Main References

- [FH] S. Fournais, B. Helffer, *Spectral Methods in Surface Superconductivity*, Progr. Nonlinear Differential Equations Appl. **77**, 2010.
- [SS] E. Sandier, S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Progr. Nonlinear Differential Equations Appl. **70**, 2007.
- [Ti] M. Tinkham, *Introduction to Superconductivity*, 2004.

Superconductivity

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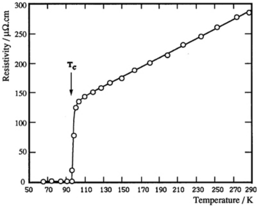
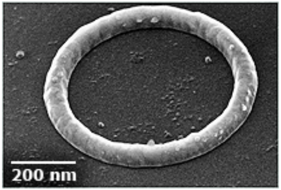
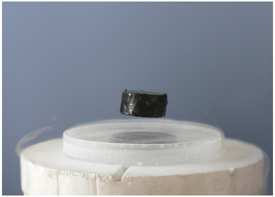
Certain materials which behave like *metals* at *room temperature* become superconductors below a certain $T_c > 0$ (ceramic compound $YBa_2Cu_3O_7$ in fig.): they show *no resistance* to the current flow, giving rise to peculiar phenomena...





Persistent currents flowing *forever* [PhysOrg.com].



Meissner effect [Wikipedia].

Maglev train running at 600 km/h [Weibo].

Response to a Magnetic Field

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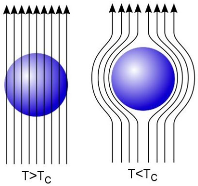
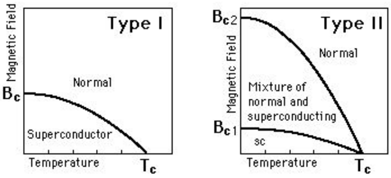
The presence of an external magnetic field might affect the behavior of superconductors, which are classified in type-I and type-II superconductors depending on their response.

Weak magnetic fields are expelled from the sample (Meissner effect): the deflection of the field generates a force able to let the sample levitate.



Meissner effect [Wikipedia].

Strong magnetic fields can penetrate the sample and eventually destroy superconductivity. The penetration occurs first at isolated points and then expands in the whole bulk of the sample.



Type-I vs Type-II [HyperPhysics].

Theory of Superconductivity

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- The microscopic model describing (conventional) superconductivity is the BCS theory, developed by Bardeen-Cooper-Schrieffer in 1957: the phenomenon is due to phase transition associated with the emergence of a collective behavior of charge carriers (electrons).
- Conducting electrons bind together to form weakly bound pairs (Cooper pairs) and such pairs undergo a sort of Bose-Einstein condensation for charged particles, i.e., all the pairs behaves the same.
- Close to the temperature marking the phase transition, a one-particle wave function is sufficient to describe the behavior of all conducting particles. This is the key idea behind the Ginzburg-Landau (GL) theory of superconductivity, which was however proposed in 1950, based on purely phenomenological considerations.

GL Theory (I)

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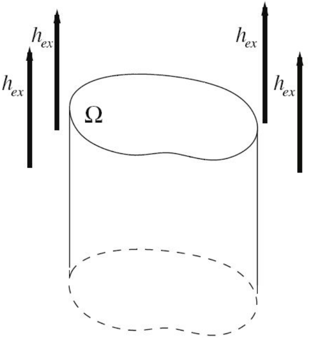
GL energy functional

The free energy per unit length is given by ($\nabla_{\mathbf{A}}\Psi := (\nabla + i\mathbf{A})\Psi$)

$$\mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r}\, \left\{ |\nabla_{\mathbf{A}}\Psi|^2 - \kappa^2|\Psi|^2 + \tfrac{1}{2}\kappa^2|\Psi|^4 \right\} + \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}\, |\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}|^2$$

- Wire of cross section $\Omega \subset \mathbb{R}^2$ bounded, smooth and simply connected (hp).
- $|\Psi|^2 : \Omega \to \mathbb{R}^+$ relative density of Cooper pairs.
- $\mathbf{A}$ magnetic potential with magnetic field $h = \mathrm{curl}\mathbf{A} = \nabla^{\perp} \cdot \mathbf{A}$.
- $\kappa \propto$ inverse of the London penetration depth ($\kappa \gg 1$ = extreme type-II).
- Uniform applied magnetic field h_{ex}.



GL Theory (II)

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- Any stable stationary configuration $(\Psi, \mathbf{A})$ of the sample is a critical point of the GL functional.
- A ground state $(\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}, \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}})$ is a lowest energy configuration (if any) of $\mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ with energy

$$E^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} := \inf_{(\psi,\mathbf{A}) \in H^1(\Omega) \times H^1(\mathbb{R}^2;\mathbb{C}^2)} \mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi, \mathbf{A}].$$

Exercise

Show that any critical point satisfies the Euler-Lagrange equations

$$\begin{cases} -\left(\nabla + i\mathbf{A}\right)^2 \Psi = \kappa^2 \left(1 - |\Psi|^2\right) \Psi, & \text{in } \Omega, \\ \nabla^{\perp} \left(\mathrm{curl}\mathbf{A}\right) = \mathbf{j}_{\mathbf{A}}[\Psi] \mathbb{1}_{\Omega}, & \text{in } \mathbb{R}^2, \\ \boldsymbol{\nu} \cdot \left(\nabla + i\mathbf{A}\right) \Psi = 0, & \text{on } \partial\Omega. \end{cases} \qquad (\star)$$

with $\boldsymbol{\nu}$ the normal to $\partial\Omega$ and $\mathbf{j}_{\mathbf{A}}[\Psi] := \Im\left[\Psi \left(\nabla - i\mathbf{A}\right) \Psi^*\right]$ (supercurrent).

Gauge Invariance

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$$\mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r}\ \left\{ \left|\nabla_{\mathbf{A}}\Psi\right|^2 - \kappa^2|\Psi|^2 + \tfrac{1}{2}\kappa^2|\Psi|^4 \right\} + \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}\ \left|\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}\right|^2$$

Proposition (Gauge invariance)

For any $\phi \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$,

$$\mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}\left[\Psi e^{-i\phi}, \mathbf{A} + \nabla\phi\right].$$

- Physically relevant quantities are gauge invariant, as, e.g., the density $|\Psi|^2$, the magnetic field $\mathrm{curl}\mathbf{A}$, the supercurrent $\mathbf{j}_{\mathbf{A}}[\Psi]$.

Proof.

- $\mathrm{curl}\nabla\phi = \nabla^\perp \cdot \nabla\phi = 0$ for any $\phi \in H^2_{\mathrm{loc}}$;
- $\left|(\nabla + i\mathbf{A} + i\nabla\phi)\left(\Psi e^{-i\phi}\right)\right|^2 = \left|(\nabla + i\mathbf{A})\,\Psi\right|^2$.

The gauge transf. $(\Psi,\mathbf{A}) \to (\Psi e^{-i\phi}, \mathbf{A} + \nabla\phi)$ leaves $\mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ invariant. □

Gauge Choice (I)

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Proposition (Coulomb gauge)

Let Ω *be as in* (hp). *Then, for any* $\mathbf{A} \in H^1(\mathbb{R}^2;\mathbb{C}^2)$, $\exists \phi \in H^2(\mathbb{R}^2;\mathbb{R})$, *such that* $\mathbf{A}_0 := \mathbf{A} + \nabla\phi$ *satisfies*

$$\begin{cases} \nabla \cdot \mathbf{A}_0 = 0, & \textit{in } \Omega, \\ \boldsymbol{\nu} \cdot \mathbf{A}_0 = 0, & \textit{on } \partial\Omega. \end{cases}$$

Proof.

The elliptic problem

$$\begin{cases} -\Delta\phi = -\nabla \cdot \mathbf{A}, & \text{in } \Omega, \\ \partial_n \phi = -\boldsymbol{\nu} \cdot \mathbf{A}, & \text{on } \partial\Omega. \end{cases}$$

admits a weak solution by Lax-Millgram theorem. The solution is also at least $H^2(\Omega)$, by elliptic regularity and smoothness of $\partial\Omega$. □

Gauge Choice (II)

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Corollary

For any $\mathbf{A}_0 \in H^1(\mathbb{R}^2)$ in the Coulomb gauge, $\exists C$ (depending only on Ω) such that

$$\left\| \mathbf{A}_0 \right\|_{H^1(\Omega)} \leqslant C \left\| \mathrm{curl} \mathbf{A}_0 \right\|_{L^2(\Omega)}, \qquad \left\| \mathbf{A}_0 \right\|_{H^2(\Omega)} \leqslant C \left\| \mathrm{curl} \mathbf{A}_0 \right\|_{H^1(\Omega)}.$$

Proof.

The differential form $\omega := (\mathbf{A}_0)_y \, \mathrm{d}x - (\mathbf{A}_0)_x \, \mathrm{d}y$ is closed and thus exact. Hence, $\exists f \in H^2(\Omega)$ such that $\omega = \mathrm{d}f$, $\boldsymbol{\nu} \cdot \nabla^{\perp} f = \boldsymbol{\nu} \cdot \mathbf{A}_0 = 0$ on $\partial\Omega$ and

$$\begin{cases} \Delta f = \mathrm{curl} \mathbf{A}_0, & \text{in } \Omega, \\ f = 0, & \text{on } \partial\Omega. \end{cases}$$

Since $\left\| \mathbf{A}_0 \right\|_{H^1(\Omega)} = \left\| \nabla f \right\|_{H^1(\Omega)}$, the result follows from elliptic theory. □

Gauge Choice (III)

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$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r} \, \left\{ \left| (\nabla + i\mathbf{A})\,\Psi \right|^2 - \kappa^2 |\Psi|^2 + \tfrac{1}{2}\kappa^2 |\Psi|^4 + \left| \mathrm{curl}\mathbf{A} - h_{\mathrm{ex}} \right|^2 \right\}$$

Proposition (Domain restriction)

Let Ω *be as in* (hp). *Then,*

$$E^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} = \inf_{(\Psi,\mathbf{A}) \in H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega;\mathbb{C}^2)} \mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}],$$

with $H^1_{\mathrm{div}}(\Omega;\mathbb{C}^2) := \left\{ \mathbf{A} \in H^1(\Omega;\mathbb{C}^2) \right.$ *in the Coulomb gauge* $\left.\right\}$. *Furthermore, given any g.s.* $(\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}, \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}})$ *in the Coulomb gauge of* $\mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$, *its restriction to* Ω *minimizes* $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$. *Conversely, any extension of a g.s. of* $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ *such that* $\mathrm{curl}\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} = h_{\mathrm{ex}}$ *in* $\mathbb{R}^2 \setminus \Omega$ *minimizes* $\mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$.

- The EL eqs. for $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ are still given by ($\star$) but additionally

$$\mathrm{curl}\mathbf{A} = h_{\mathrm{ex}}, \qquad \text{on } \partial\Omega.$$

$$\mathcal{G}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] - \mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \int_{\mathbb{R}^2\setminus\Omega} \mathrm{d}\mathbf{r}\ \left|\mathrm{curl}\mathbf{A} - h_{\mathrm{ex}}\right|^2$$

Proof.

The restriction of any $(\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}, \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}})$ in the Coulomb gauge to Ω yields the inequality $\inf_{(\Psi,\mathbf{A})\in H^1(\Omega)\times H^1_{\mathrm{div}}(\Omega;\mathbb{C}^2)} \mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] \leqslant E^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$.

For any given $(\Psi,\mathbf{A}) \in H^1(\Omega)\times H^1_{\mathrm{div}}(\Omega;\mathbb{C}^2)$, we need to find an extension $\widetilde{\mathbf{A}} \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ of $\mathbf{A}$ such that $\mathrm{curl}\widetilde{\mathbf{A}} = h_{\mathrm{ex}}$ in $\mathbb{R}^2\setminus\Omega$: consider $\mathbf{B} = \nabla^{\perp} f$ solving

$$\begin{cases} -\Delta f = -\mathrm{curl}\mathbf{A}, & \text{in } \Omega, \\ -\Delta f = -h_{\mathrm{ex}}, & \text{in } \mathbb{R}^2\setminus\Omega. \end{cases}$$

which is solved by $f = (-\Delta + h_{\mathrm{ex}})^{-1} g$, where $(-\Delta+\lambda)^{-1}$ is the resolvent of the Laplacian in $\mathbb{R}^2$ and $g = (-\mathrm{curl}\mathbf{A} + h_{\mathrm{ex}})\,\mathbb{1}_{\Omega}$. Then, $\mathrm{curl}\mathbf{B} = \mathrm{curl}\mathbf{A}$ in Ω and thus $\mathbf{B} - \mathbf{A} = \nabla\phi$. Setting $\widetilde{\mathbf{A}} = \mathbf{B} - \nabla\phi$, we get the result. □

Ground States

$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r}\ \left\{ \left| (\nabla + i\mathbf{A})\,\Psi \right|^2 - \kappa^2 |\Psi|^2 + \tfrac{1}{2}\kappa^2 |\Psi|^4 + \left| \mathrm{curl}\mathbf{A} - h_{\mathrm{ex}} \right|^2 \right\}$$

Proposition (Existence of ground states)

Let Ω be as in (hp). Then, $\exists$ a smooth ground state $(\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}, \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}})$ (in the Coulomb gauge) of $\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$.

Proof (I).

By completing the square and testing on $(1,0)$, we get

$$-\tfrac{1}{2}\kappa^2 \left|\Omega\right| \leqslant E^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \leqslant \tfrac{1}{2} h_{\mathrm{ex}}^2 \left|\Omega\right|.$$

Any minimizing sequence $\{(\Psi_n, \mathbf{A}_n)\}_{n\in\mathbb{N}}$ satisfies $\left\| 1 - |\Psi_n|^2 \right\|_{L^2(\Omega)} \leqslant C$,

$$\left\| (\nabla + i\mathbf{A}_n)\,\Psi_n \right\|_{L^2(\Omega)} \leqslant C, \qquad \left\| \mathbf{A}_n \right\|_{H^1(\Omega)} \leqslant C \left\| \mathrm{curl}\mathbf{A}_n \right\|_{L^2(\Omega)} \leqslant C,$$

$$\mathcal{F}^{\rm GL}_{\kappa,h_{\rm ex}}[\Psi,\mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r} \, \left\{ \left| (\nabla + i\mathbf{A})\,\Psi \right|^2 - \kappa^2 |\Psi|^2 + \tfrac{1}{2}\kappa^2 |\Psi|^4 + \left| \mathrm{curl}\mathbf{A} - h_{\rm ex} \right|^2 \right\}$$

Proof (II).

We deduce that $\exists(\Psi^{\rm GL}, \mathbf{A}^{\rm GL}) \in H^1(\Omega) \times H^1_{\rm div}(\Omega;\mathbb{C}^2)$ such that

$$\Psi_n \xrightarrow[n\to+\infty]{\mathrm{w}-H^1(\Omega)} \Psi^{\rm GL}, \qquad \mathbf{A}_n \xrightarrow[n\to+\infty]{\mathrm{w}-H^1_{\rm div}(\Omega;\mathbb{C}^2)} \mathbf{A}^{\rm GL}.$$

By Rellich-Kondrashev theorem, $\Psi_n \xrightarrow[n\to+\infty]{L^p(\Omega)} \Psi^{\rm GL}$, $1 \leqslant p < +\infty$ and, by lower semicontinuity of the norms,

$$\liminf_{n\to\infty} \left\| \nabla_{\mathbf{A}_n} \Psi_n \right\|_{L^2(\Omega)} \geqslant \left\| \nabla_{\mathbf{A}^{\rm GL}} \Psi^{\rm GL} \right\|_{L^2(\Omega)},$$

$$\liminf_{n\to\infty} \left\| \mathrm{curl}\mathbf{A}_n - h_{\rm ex} \right\|_{L^2(\Omega)} \geqslant \left\| \mathrm{curl}\mathbf{A}^{\rm GL} - h_{\rm ex} \right\|_{L^2(\Omega)},$$

so that $\mathcal{F}^{\rm GL}_{\kappa,h_{\rm ex}}[\Psi^{\rm GL}, \mathbf{A}^{\rm GL}] = E^{\rm GL}_{\kappa,h_{\rm ex}}$.
Smoothness follows from elliptic regularity of the GL equations ($\star$). □

Maximum Principle

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Proposition (Maximum principle)

Let Ω be as in (hp). Then, any $(\Psi, \mathbf{A})$ solving ($\star$) satisfies

$$\|\Psi\|_{L^\infty(\Omega)} \leqslant 1.$$

Exercise

Prove that, if (hp) holds, any weak solution of ($\star$) is actually smooth in $\overline{\Omega}$.

Proof.

The first equation in ($\star$) $-(\nabla + i\mathbf{A})^2 \Psi = \kappa^2(1 - |\Psi|^2)\Psi$, together with $\frac{1}{2}\Delta|\Psi|^2 = |\nabla\Psi|^2 + \Re(\Psi^*\Delta\Psi)$, yields

$$-\tfrac{1}{2}\Delta|\Psi|^2 \leqslant \kappa^2(1 - |\Psi|^2)\,|\Psi|^2\,.$$

At any maximum point of $|\Psi|^2$, $\Delta|\Psi|^2 \leqslant 0$. □

Conclusions

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- We can restrict the integration domain to Ω and consider
$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r}\ \left\{ \left| (\nabla + i\mathbf{A})\,\Psi \right|^2 - \kappa^2 |\Psi|^2 + \tfrac{1}{2}\kappa^2 |\Psi|^4 + \left| \mathrm{curl}\mathbf{A} - h_{\mathrm{ex}} \right|^2 \right\}$$
- The GL functional is gauge invariant:
$$\mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi,\mathbf{A}] = \mathcal{F}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}[\Psi e^{-i\phi}, \mathbf{A} + \nabla\phi].$$
- There exists at least one ground state $(\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}, \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}})$ such that
  - any $(\Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}, \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}})$ is smooth;
  - $\left\| \Psi^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \right\|_\infty \leqslant 1$;
  - $\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}}$ is in the Coulomb gauge:
$$\begin{cases} \nabla \cdot \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} = 0, & \text{in } \Omega, \\ \boldsymbol{\nu} \cdot \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} = 0, & \text{on } \partial\Omega. \end{cases}$$

    and therefore $\left\| \mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \right\|_{H^1(\Omega)} \leqslant C \left\| \mathrm{curl}\mathbf{A}^{\mathrm{GL}}_{\kappa,h_{\mathrm{ex}}} \right\|_{L^2(\Omega)}$.