Emergence of Haldane pseudopotentials in systems with short range interactions

Jakob Yngvason, University of Vienna

with Robert Seiringer, IST Austria

J. Stat. Phys. **281**, 448–464 (2020)

Gran Sasso Science Institute, March 24, 2021

The Quantum Hall Effect

The integer quantum Hall effect (IQHE) is a quantum phenomenon in strong magnetic fields. The Hall conductance of 2D electrons (fermions) shows precisely quantized plateaus as function of the filling factor (density/field strength) at integer values of $\frac{e^2}{h}$. (Nobel prize 1985: von Klitzing)

In the fractional quantum Hall effect (FQHE) the Hall conductance shows precisely quantised plateaus at fractional values of $\frac{e^2}{h}$. (Nobel prize 1998: Laughlin, Störmer, Tsu; 2016: Haldane.)

A bosonic FQHE is theoretically possible in rapidly rotating cold atomic gases.

The microscopic origin of the FQHE is a major research topic in condensed matter physics.

Magnetic Hamiltonian

The 2D Hamiltonian in a constant magnetic field of strength B and units so that $\hbar = 1$, charge=1, mass $=\frac{1}{2}$, is

$$H_{\text{magn}} = -\left(\nabla_\perp + \mathrm{i}\,\mathbf{A}(\mathbf{r})\right)^2$$

with $\mathbf{A}(\mathbf{r}) = \frac{B}{2}(-y, x)$.
In complex coordinates $z = x + \mathrm{i}y$ and units so that $B = 2$ it can be written

$$\boxed{H_{\text{magn}} = 4\left(\hat{a}^\dagger \hat{a} + \tfrac{1}{2}\right)}$$

with

$$\boxed{\hat{a} = \left(\bar{\partial} + \tfrac{1}{2}z\right), \qquad \hat{a}^\dagger = \left(-\partial + \tfrac{1}{2}\bar{z}\right), \qquad [a, a^\dagger] = 1.}$$

The spectrum of H_{magn} is $\varepsilon_n = 4(n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$ (Landau levels).

A rotating system can be described by the same Hamiltonian with B replaced by 2 times the rotational velocity.

Lowest Landau Level

The lowest Landau level in $L^2(\mathbb{R}^2)$ is characterized by $\hat{a}\psi = 0$, i.e.,

$$\bar{\partial}\psi = -\tfrac{1}{2}z\,\psi.$$

This is the Cauchy-Riemann equation for $\psi(z)e^{|z|^2/2}$ with general solution

$$\psi(z) = f(z)e^{-|z|^2/2}$$

with an *analytic* function f.

For an N-particle system the lowest Landau level (LLL) in $\mathcal{H} := L^2(\mathbb{R}^{2N})$ consisting of functions

$$\boxed{\Psi(x_1,\dots,x_N) = \psi(z_1,\dots,z_N)\prod_{i=1}^{N}\mathrm{e}^{-|x_i|^2/2}}$$

with $\psi : \mathbb{C}^N \to \mathbb{C}$ analytic. For bosons resp. fermions the $\psi(z_1,\dots,z_N)$ are symmetric resp. antisymmetric.

Interactions. Laughlin wave functions

To study the effects of two-body interactions in the the FQHE, Duncan Haldane introduced in 1983 interaction operators (pseudopotentials) which act on N-particle wave functions in the lowest Landau level (LLL) and have the Laughlin wave functions as exact ground states.

The Laughlin wave functions are (with m a positive integer, even for bosons, odd for fermions)

$$\Psi_m^{\mathrm{L}}(x_1, \ldots, x_N) = C_{N,m} \prod_{1 \leq i < j \leq N} (z_i - z_j)^m \prod_{i=1}^{N} \mathrm{e}^{-|x_i|^2/2}$$

They have filling fraction $1/m$.

The Pseudopotential Operators

Haldane expands, in the LLL, a (sufficiently smooth!) radially symmetric two body interaction $v(x_i - x_j)$ into a sum of projections on angular momentum eigenstates in the difference variables:

$$P_{\rm LLL} v(x_i - x_j) P_{\rm LLL} = \sum_{\ell \geq 0} \langle \varphi_\ell | v | \varphi_\ell \rangle \mathfrak{P}_{i,j}^{(\ell)} , \quad \varphi_\ell(z) = (\pi \ell!)^{-1/2} z^\ell e^{-|x|^2/2}$$

with the Haldane pseudo-potential operators

$$\boxed{\sum_{1 \leq i < j \leq N} \mathfrak{P}_{ij}^{(\ell)}}$$

as effective interaction Hamiltonians. They have the Laughlin wave functions $\Psi_m^{\rm L}$ as exact ground states for $m > \ell$.

The Pseudopotential Operators (cont.)

More precisely:

An analytic function of $z_1, \dots, z_N$ can be expanded in powers of the difference variable $z_i - z_j$ with $z_i + z_j$ and z_k, $k \neq i, j$, fixed.

As a projection operator on LLL $\mathfrak{P}_{i,j}^{(\ell)}$ picks out the ℓ-th term in this expansion of the analytic factor of the wave function, annihilating the other terms. Hence $\mathfrak{P}_{i,j}^{(\ell)}$ can be regarded as a zero-range interaction.

In fact, as a quadratic form on states in the LLL with relative angular momentum $\geq \ell$ in the variables (x_i, x_j), $\mathfrak{P}_{i,j}^{(\ell)}$ is formally equal to

$$\boxed{2\pi(2^\ell \ell!)^{-1} \Delta^\ell \delta(x_i - x_j)}$$

with $\Delta = \nabla^2$ the Laplacian.

The Objective

For strong, short-ranged interactions (e.g. hard balls) the simple expansion into pseudo-potentials is clearly not viable; the “moments” $\langle\varphi_\ell|v|\varphi_\ell\rangle \sim \int v(r)r^\ell e^{-r^2}dx$ need not even be defined.

Goal:

Derive the effective low-energy description in the LLL of a $2D$ system of particles with strong, short-range interactions.

It turns out that the contributions of higher Landau levels to the interaction lead to a renormalization of the pre-factors to the Haldane pseudo-potentials. Effectively, the moments $\langle\varphi_\ell|v|\varphi_\ell\rangle$ need to be replaced by factors involving the ℓ-wave scattering lengths of v.

Our work is a natural generalization of [M. Lewin, R. Seiringer, JSP 137, 1040 (2009)] where only the lowest ℓ sector in the bosonic case was considered.

The Model

On $L^2(\mathbb{R}^2)$, let h denote the 1-particle magnetic Laplacian (for $B = 2$), with its ground state energy shifted to zero, having spectrum $4\mathbb{N} \cup \{0\}$:

$$h = (-\mathrm{i}\nabla - x^\perp)^2 - 2$$

where $x^\perp = (-x^{(2)}, x^{(1)})$ for $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^2$. Its kernel is the LLL:

$$\mathrm{span}\left\{ z^\ell \mathrm{e}^{-|z|^2/2},\ \ell \geq 0,\ z = x^{(1)} + \mathrm{i}x^{(2)} \right\}.$$

For $v \geq 0$ radial and of compact support, and $a > 0$, define the N-particle Hamiltonian

$$H_a = \sum_{i=1}^{N} h_i + \sum_{1 \leq i < j \leq N} v_a(|x_i - x_j|)\ , \quad v_a(r) = a^{-2} v(r/a)$$

on $\mathcal{H} = L^2(\mathbb{R}^{2N})$. We are interested in the regime $a \ll 1$.

The Model (cont.)

Remarks

- To motivate the scaling of the potential note that the Laplacian scales like the square of an inverse length. The parameter a can be thought of as the ratio of the s-wave scattering length of v_a to the magnetic length (which is $O(1)$ in our units).
- As $a \to 0$, v_a may create states in arbitrarily high LL. Although we are eventually interested in the LLL it is essential to keep the full kinetic energy operator in all LL to control the short range effects the potential v_a has on the spectrum in the LLL .
- The fact that H_a leads out of the LLL is reflected in the type of convergence we consider: It is natural to use convergence of resolvents, which suppress state components in higher LL.

The spaces $\mathfrak{B}_\ell$

We introduce the sequence of closed subspaces

$$\mathcal{H} \supset \mathfrak{B}_0 \supset \mathfrak{B}_1 \supset \dots$$

where $\mathfrak{B}_\ell$ for $\ell \geq 0$ consists of $\Psi \in \mathcal{H}$ of the form

$$\Psi(x_1, \dots, x_N) = \varphi(z_1, \dots, z_N) \prod_{i<j} (z_i - z_j)^\ell \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^N |x_i|^2}$$

with $\varphi : \mathbb{C}^N \to \mathbb{C}$ analytic.

In particular $\mathfrak{B}_0$=LLL , and for $\ell \geq 1$ the functions in

$$\mathfrak{B}_\ell = \{\varphi \Psi_\ell^{\mathrm{L}} | \, \varphi \text{ analytic}\}$$

have all relative angular momenta $\geq \ell$.

The Effective Hamiltonians

On $\mathfrak{B}_\ell$, we define the operators

$$\boxed{\mathfrak{h}_\ell = \sum_{i<j} \mathfrak{D}_{ij}^{(\ell)}}$$

where

$$\boxed{\langle \Psi | \mathfrak{D}_{12}^{(\ell)} \Psi \rangle = \int_{\mathbb{R}^2} \mathrm{e}^{-2|x|^2} |\varphi(z,z)|^2 dx}$$

for

$$\Psi(x_1, x_2) = \varphi(z_1, z_2)(z_1 - z_2)^\ell \mathrm{e}^{-\frac{1}{2}(|x_1|^2 + |x_2|^2)}.$$

By comparing matrix elements one sees that $\mathfrak{D}_{ij}^{(\ell)} = 1/(\pi \ell!) \mathfrak{P}_{ij}^{(\ell)}$ so

$$\boxed{\mathfrak{h}_\ell = \frac{1}{\pi \ell!} \sum_{i<j} \mathfrak{P}_{ij}^{(\ell)}}$$

Also note that the kernel of $\mathfrak{h}_\ell$ coincides with $\mathfrak{B}_{\ell+1}$, the domain of $\mathfrak{h}_{\ell+1}$.

The Scattering Length(s)

For $\ell \geq 1$, the relevant scattering parameters in arbitrary angular momentum channels are given by the variational principle

$$b_\ell = \frac{1}{4\pi\ell} \min \left(\int_{\mathbb{R}^2} |x|^{2\ell} \left[|\nabla f(x)|^2 + \tfrac{1}{2} v(|x|) |f(x)|^2 \right] dx \; : \; \lim_{|x|\to\infty} f(x) = 1 \right\} .$$

One has $b_\ell = R_0^{2\ell}$ for hard discs of diameter R_0. (For $\ell = 0$ the definition of the scattering length has to be modified slightly in 2D because the scattering solution increases logarithmically.)
The variational equation (zero-energy scattering equation in the ℓ channel) for the minimizer reads

$$-f_\ell''(r) - (2\ell + 1) r^{-1} f_\ell'(r) + \tfrac{1}{2} v(r) f_\ell(r) = 0.$$

If v is replaced by v_a then b_ℓ is replaced by $a^{2\ell} b_\ell$ for $\ell \geq 1$ and b_0 by ab_0.

The Main Result

Recall that

$$H_a = \sum_{i=1}^{N} h_i + \sum_{1 \le i < j \le N} v_a(|x_i - x_j|) \ , \quad v_a(r) = a^{-2} v(r/a)$$

Our main result says that for small a,

$H_a \approx a^{2\ell}\, 8\pi\ell\, b_\ell\, \mathfrak{h}_\ell$ for energies of the order $a^{2\ell}$. To be precise:

Theorem

For any $\ell \geq 1$, the operator $a^{-2\ell} H_a$ converges to $8\pi\ell b_\ell \mathfrak{h}_\ell$ in strong resolvent sense as $a \to 0$, i.e., for any $\mu > 0$ and $\Psi \in \mathcal{H} = L^2(\mathbb{R}^{2N})$

$$\lim_{a \to 0} (\mu + a^{-2\ell} H_a)^{-1} \Psi = (\mu + 8\pi\ell b_\ell \mathfrak{h}_\ell)^{-1} P_\ell \Psi$$

strongly in $L^2(\mathbb{R}^{2N})$, where P_ℓ denotes the projection onto $\mathfrak{B}_\ell \subset \mathcal{H}$. Moreover, for $\ell = 0$, $\ln(1/a^2) H_a$ converges in the same sense to $8\pi \mathfrak{h}_0$.

Remarks

- The effective Hamiltonians are not given by simply projecting H_a onto the subspace $\mathfrak{B}_\ell$. In fact

$$P_\ell H_a P_\ell \approx a^{2\ell} \int_{\mathbb{R}^2} v(r) r^{2\ell} e^{-r^2} dx \, \mathfrak{h}_\ell$$

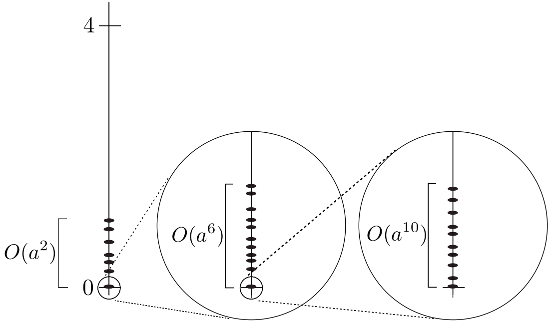
i.e., $\int v r^{2\ell} e^{-r^2}$ appears in place of $8\pi\ell b_\ell$. The short-scale structure of the eigenfunctions of H_a leads to an effective renormalization of the prefactors.

- An analogous result holds in $3D$, with an additional confinement in the third direction. One considers a potential $V : \mathbb{R} \to \mathbb{R}$ such that the Schrödinger operator $-\partial_u^2 + V(u)$ has a ground state $\chi \in H^1(\mathbb{R})$ with corresponding energy E, and on $L^2(\mathbb{R}^3)$

$$h = (-\mathrm{i}\nabla + x \wedge e_3)^2 + V(x^{(3)}) - 2 - E$$

One then shows that $a^{-2\ell-1} H_a$ converges to $8\pi(2\ell+1)\tilde{b}_\ell \int |\chi|^4 \, \mathfrak{h}_\ell$

Schematic Picture



Sketch of the spectrum in the fermionic case. States with energy of order a^2 are described by the effective Hamiltonian $\mathfrak{h}_1$ on the LLL. Zooming in on its kernel, one finds $\mathfrak{h}_3$ as an effective Hamiltonian, describing states with energy of order a^6, and so forth.

The scaling of H_a by $a^{-2\ell}$ acts as a magnification glas!

Ground states

The convergence Theorem can be generalized to

$$a^{-2\ell} H_a + K \xrightarrow{a\to 0} 8\pi\ell b_\ell\, \mathfrak{h}_\ell + P_\ell K P_\ell$$

for bounded, and also suitable unbounded operators K. If K is a confining external potential, the convergence holds in norm resolvent sense, which implies convergence of eigenvalues and eigenvectors. In particular, one has for $K = \sum_{i=1}^N |x_i|^2$

Corollary

Fix $\ell \geq 1$. For $\lambda > 0$ small enough, the ground state of $H_a + a^{2\ell}\lambda \sum_{i=1}^N |x_i|^2$ converges in $L^2(\mathbb{R}^{2N})$ to the Laughlin state $\Psi^{\mathrm{L}}_{\ell+1}$ as $a \to 0$.

For $\ell = 0$ this holds with $a^{2\ell}$ replaced by $(\ln(1/a^2))^{-1}$.

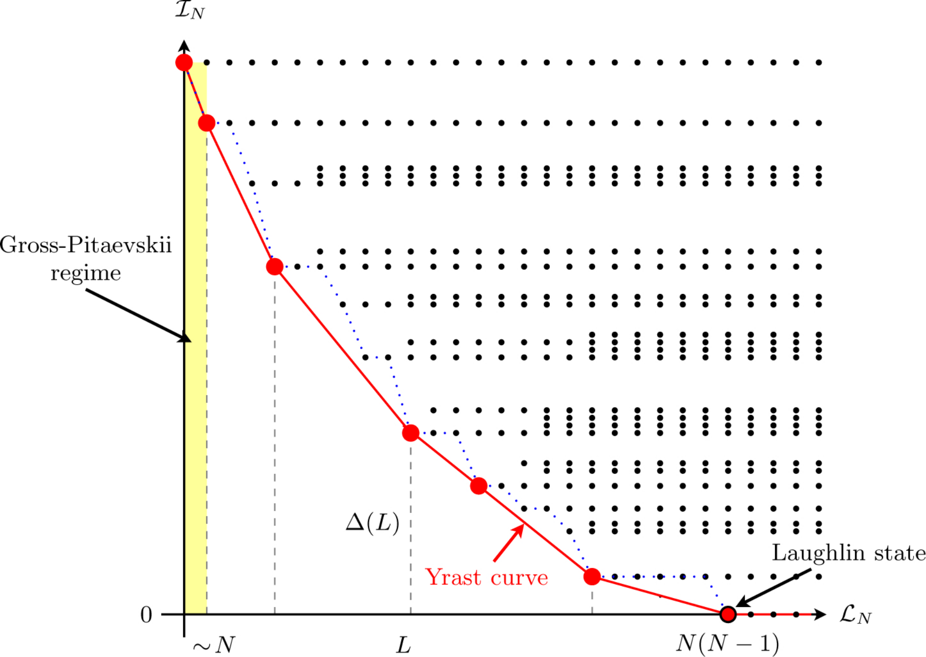
Yrast Curves

In the LLL a quadratic confining term is, up to a constant, equivalent to the angular momentum operator.

Define $\mathcal{L} := P_\ell(\sum_{i=1}^N |x_i|^2)P_\ell$ and consider for $\gamma, \lambda > 0$ the ground state(s) of

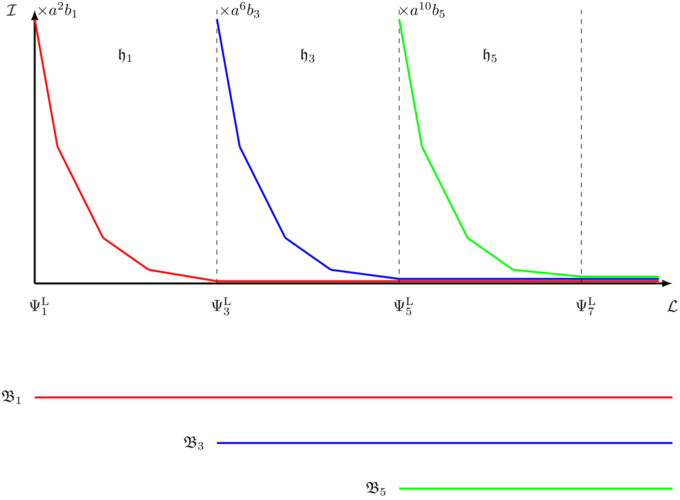
$$\gamma\mathfrak{h}_\ell + \lambda\mathcal{L}.$$

The operators $\mathfrak{h}_\ell$ and $\mathcal{L}$ commute and thus have a joint spectrum. The boundary of its convex hull is called the **Yrast curve** for $\mathfrak{h}_\ell$. The ground state of $\gamma\mathfrak{h}_\ell + \lambda\mathcal{L}$ is determined by the point(s) where a line of slope $(-\lambda/\gamma)$ touches the Yrast curve.



$\mathcal{I}_N$
Gross-Pitaevskii
regime
$\Delta(L)$
Yrast curve
Laughlin state
0
$\mathcal{L}_N$
$\sim N$
L
$N(N-1)$

Yrast Curves for fermions



Gamma Convergence

Strong resolvent convergence of operators is equivalent to Γ-convergence of the corresponding quadratic forms in both weak and strong topologies. The proof of the Main Theorem is based on this fact.

For $\Psi \in \mathcal{H}$ define

$$F_a(\Psi) := \|\sqrt{H_a}\Psi\|^2$$

(with $F_a = \infty$ if Ψ is not in the form domain of H_a),

and, for $\ell \geq 0$,

$$G^{(\ell)}(\Psi) := \begin{cases} \langle \Psi | \mathfrak{h}_\ell \Psi \rangle & \text{if } \Psi \in \mathfrak{B}_\ell \\ +\infty & \text{if } \Psi \notin \mathfrak{B}_\ell \end{cases}$$

Gamma Convergence (cont.)

Proposition

For any $\ell \geq 1$, the function $a^{-2\ell} F_a$ Γ-converges to $8\pi \ell b_\ell G^{(\ell)}$ as $a \to 0$ both in the strong and weak topology on $\mathcal{H}$, i.e.,

1. *if $\lim_{n\to\infty} a_n = 0$ and $\Psi_n \rightharpoonup \Psi \in \mathcal{H}$ as $n \to \infty$, then*

$$\liminf_{n\to\infty} a_n^{-2\ell} F_{a_n}(\Psi_n) \geq 8\pi \ell b_\ell G^{(\ell)}(\Psi) .$$

2. *for any $\Psi \in \mathcal{H}$ and any sequence $\{a_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a_n = 0$, there exists a sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ with $\Psi_n \to \Psi \in \mathcal{H}$ as $n \to \infty$ such that*

$$\lim_{n\to\infty} a_n^{-2\ell} F_{a_n}(\Psi_n) \leq 8\pi \ell b_\ell G^{(\ell)}(\Psi) .$$

For $\ell = 0$, the same holds with $a^{-2\ell}$ replaced by $\ln(1/a^2)$, and ℓb_ℓ replaced by 1.

The Lower Bound is Based on a Dyson Lemma

A “Dyson Lemma” (Dyson 1957) allows to replace, for the purpose of a lower bound, a strong short range potential by a weak long range potential by sacrifying the kinetic energy. In the present context it takes the following form:

Lemma (Dyson Lemma)

For any $\ell \geq 1$ and $R > aR_0$, and any $y \in \mathbb{C}$, we have for all $\varphi : \mathbb{R}^2 \to \mathbb{C}$

$$\int_{|z-y|<R} \mathrm{e}^{-|x|^2} \left(4|\partial_{\bar{z}}\varphi(x)|^2 + \tfrac{1}{2}v_a(|z-y|)|\varphi(x)|^2\right) dx$$

$$\geq 4\pi\ell b_\ell a^{2\ell} \mathrm{e}^{|y|^2-R^2} \left| \frac{1}{2\pi\mathrm{i}} \oint_{|z-y|=R} \mathrm{e}^{-z\bar{y}} \frac{\varphi(x)}{(z-y)^{\ell+1}} dz \right|^2$$

Dyson Lemma (cont.)

Note that for any $\Psi \in L^2(\mathbb{R}^2)$ of the form $\Psi(x) = \mathrm{e}^{-|x|^2/2}\varphi(x)$, we have

$$4\int_{\mathbb{R}^2} \mathrm{e}^{-|x|^2} \left|\partial_{\bar z}\varphi(x)\right|^2 dx = \langle \Psi | h\Psi \rangle .$$

By averaging over R one gets as corollary of the Lemma

$$\begin{aligned} &\int_{|z-y|<R} \mathrm{e}^{-|z|^2} \left(4|\partial_{\bar z}\varphi(x)|^2 + \tfrac{1}{2} v_a(|z-y|)|\varphi(x)|^2\right) dx \\ &\geq 4\pi\ell b_\ell a^{2\ell} \mathrm{e}^{|y|^2} \int_0^\infty \mathrm{e}^{-r^2} \rho(r) \left| \frac{1}{2\pi \mathrm{i}} \oint_{|z-y|=r} \mathrm{e}^{-z\bar y} \frac{\varphi(x)}{(z-y)^{\ell+1}} dz \right|^2 dr \qquad (1) \end{aligned}$$

with $\rho(r) = 2/R$ for $R/2 < r < R$, and 0 otherwise. In this form the lemma is used to prove the lower bound, taking $R \to 0$.

Upper Bound

For the upper bound we modify $\Psi \in \mathfrak{B}_\ell$ by introducing a correlation structure: For $a_n \to 0$ consider the sequence

$$\Psi_n(x_1, \dots, x_N) = \Psi(x_1, \dots, x_N) \prod_{1 \le i < j \le N} f_\ell((x_i - x_j)/a_n)$$

with f_ℓ the minimizer in the variational principle

$$b_\ell = \frac{1}{4\pi\ell} \min \left\{ \int_{\mathbb{R}^2} |x|^{2\ell} \left(|\nabla f(x)|^2 + \tfrac{1}{2} v(|x|) |f(x)|^2 \right) dx \ : \ \lim_{|x| \to \infty} f(x) = 1 \right\}.$$

By dominated convergence, $\Psi_n \to \Psi$ as $a_n \to 0$.

Upper Bound (cont.)

Moreover,

$$F_{a_n}(\Psi_n) = \int_{\mathbb{R}^{2N}} |\Psi(x_1,\dots,x_N)|^2 \left[4\sum_{i=1}^{N} |\partial_{\bar{z}_i} S|^2 + \tfrac{1}{2}\sum_{i<j} v_{a_n}(|x_i - x_j|)|S|^2 \right].$$

with

$$S(x_1,\dots,x_N) = \prod_{i<j} f_\ell((x_i - x_j)/a_n).$$

Since S is real, $|\partial_{\bar{z}_i} S| = \frac{1}{2}|\nabla_i S|$. By noting that

$$\mathfrak{D}_{ij}^{(\ell)} = (\bar{z}_i - \bar{z}_j)^{-\ell} \mathfrak{D}_{ij}^{(0)} (z_i - z_j)^{-\ell}$$

one can use previous results of Lewin and Seiringer for the $\ell = 0$ case to finish the upper bound:

$$\limsup_{n\to\infty} a_n^{-2\ell} F_{a_n}(\Psi_n) \le 4\pi\ell b_\ell \sum_{i\ne j} \langle \Psi | \mathfrak{D}_{ij}^{(\ell)} \Psi \rangle = 8\pi\ell b_\ell\, G^{(\ell)}(\Psi).$$

Summary

- We have rigorously derived Haldane's pseudo-potential operators for many-body quantum systems with short-range interactions subject to a (homogeneous) magnetic field, or to rapid rotation.
- In particular, we show that Laughlin's wavefunctions emerge as limits of ground states of such models under suitable confinement.
- Compared to a naive projection onto the LLL, the prefactors of the pseudo-potentials are renormalized by contributions from higher LL, and involve the scattering lengths instead of the moments of the interaction potential.

Open Problems

- It remains an open problem to quantify the N-dependence in our estimates.
- Haldane conjectured that $\mathfrak{h}_\ell$ has a spectral gap above zero that is bounded below independently of N, which has important consequences concerning stability and incompressibility.

  This as a very difficult problem! Main reason: $\mathfrak{h}_\ell$ is a sum of $\sim N^2$ noncommuting terms, because $[\mathfrak{P}_{ij}^{(\ell)}, \mathfrak{P}_{ik}^{(\ell)}] \neq 0$ if $j \neq k$.

  A one-dimensional version of the problem is easier, but still highly nontrivial. (B. Nachtergaele, S. Warzel, A. Young, arXiv:2004.04992.)