Strict deformation quantization: a C^*-algebraic approach to the classical limit of quantum systems

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Transition between quantum and classical theories

- Quantum mechanics → classical mechanics.
- Statistical mechanics of a quantum spin system → classical thermomechanics of a spin system.

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Main focus

- **Classical limit** of quantum spin systems & Schrödinger operators.
  ▷ Prof. Valter Moretti (University of Trento)
  ▷ Dr. Simone Murro (University of Paris-Saclay)
- **Emergence**, e.g. Spontaneous Symmetry Breaking (SSB).
  ▷ Prof. Klaas Landsman (Radboud University Nijmegen)
  ▷ Dr. Robin Reuvers (University of Rome 3)

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Approach

Strict deformation quantization

Mathematical approach

Quantization theory

- **Strict deformation quantization** $\rightarrow$ transition between quantum and classical theories.
- Quantization of a Poisson manifold (i.e. phase space) into a quantum (noncommutative) C*-algebra.
  (1) Continuous bundle of C^*-algebras $(A_\hbar)_{\hbar \in I}$ over base space I with $A_0 = C_0(X)$;
  (2) A dense Poisson subalgebra $\tilde{A}_0 \subset C^\infty(X) \subset A_0$;
  (3) Quantization maps $Q_\hbar : \tilde{A}_0 \rightarrow A_\hbar$;
  (4) For all $f, g \in \tilde{A}_0$ one has the Dirac-Groenewold-Rieffel condition:

$$\lim_{\hbar \to 0} || \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\})||_\hbar = 0.$$

- Rigorous notion of the 'classical limit': a precise way to describe the transition from quantum to classical theory in terms of C^*-algebras.
- Applications to spontaneous symmetry breaking (SSB) applied to quantum (spin) systems.
- C^*-algebraic approach $\rightarrow$ computations often easy to handle.

- Consider

$$A_0 = C_0(\mathbb{R}^{2n}) \ (\hbar = 0);$$
$$A_\hbar = B_\infty(L^2(\mathbb{R}^n)) \ (\hbar > 0),$$

where $\mathbb{R}^{2n}$ is equipped with thet standard symplectic Poisson structure.

$\Rightarrow$ A_0 and $A_\hbar$ form fibers of a continuous bundle of C^*- algebras over $I = [0, 1]$.

- Quantization maps: for any $\hbar \in (0, 1]$ define

$$Q_\hbar : C_c^\infty(\mathbb{R}^{2n}) \to B_\infty(L^2(\mathbb{R}^n));$$
$$Q_\hbar(f) = \int_{\mathbb{R}^{2n}} \frac{d^n p d^n q}{(2\pi\hbar)^n} f(p, q)|\phi_\hbar^{(p,q)}\rangle\langle\phi_\hbar^{(p,q)}|,$$

where for each $\hbar \in I$ the operator $|\phi_\hbar^{(p,q)}\rangle\langle\phi_\hbar^{(p,q)}|$ is the projection onto the subspace spanned by the unit vector $\phi_\hbar^{(p,q)} \in L^2(\mathbb{R}^n)$, also called a Schrödinger coherent state:

$$\phi_\hbar^{(p,q)}(x) = (\pi\hbar)^{-n/4} e^{-ipq/2\hbar} e^{-ipx/\hbar} e^{-(x-q)^2/2\hbar}. \quad (1)$$

- Consider

$$A'_0 = C(S^2),\ (1/N = 0);$$
$$A'_{1/N} = M_{N+1}(\mathbb{C}),\ (1/N > 0).$$

$\Rightarrow$ A'_0 and $A'_{1/N}$ form fibers of a continuous bundle of C^*- algebras over $I = 1/\mathbb{N} \cup \{0\}$.

- Poisson structure on S^2: $\{f, g\}(x) = \sum_{a,b,c=1}^{3} \epsilon_{abc} x_c \frac{\partial f}{\partial x_a} \frac{\partial g}{\partial x_b}$ $(x \in S^2)$, with f, g restrictions of smooth functions to S^2 $\rightarrow$ dense subspace $\tilde{A}'_0 \subset A'_0$ made of polynomials in three real variables restricted to S^2.

- Quantizations maps: for any $1/N \in 1/\mathbb{N}$:

$$Q'_{1/N} : \tilde{A}'_0 \rightarrow M_{N+1}(\mathbb{C});$$
$$Q'_{1/N}(p) = \frac{N+1}{4\pi} \int_{S^2} d\mu(\Omega) p(\Omega) |v^{(\Omega)}\rangle\langle v^{(\Omega)}|.$$

$|v^{(\Omega)}\rangle\langle v^{(\Omega)}|$ is the projection onto the linear span of the vector $v_N^{(\Omega)}$, called a N coherent spin state:

$$|v^{(\Omega)}\rangle_N = \underbrace{|v^{(\Omega)}\rangle_1 \otimes \cdots \otimes |v^{(\Omega)}\rangle_1}_{N \text{ times}} \in (\mathbb{C}^2)^{\otimes_s N} \equiv \mathrm{Sym}^N(\mathbb{C}^2) \cong \mathbb{C}^{N+1},$$

where $|v^{(\Omega)}\rangle_1 = \cos\frac{\theta_\Omega}{2}|\uparrow\rangle + e^{i\phi_\Omega}\sin\frac{\theta_\Omega}{2}|\downarrow\rangle \in \mathbb{C}^2$ and $|\uparrow\rangle, |\downarrow\rangle$ are the eigenvectors of σ_3 in $\mathbb{C}^2$, so that $\sigma_3|\uparrow\rangle = |\uparrow\rangle$ and $\sigma_3|\downarrow\rangle = -|\downarrow\rangle$.

Study: classical limit of quantum theories and SSB

• Study: asymptotic properties of vectors in Hilbert space $\mathcal{H}_\hbar$; e.g. think of eigenvectors $(\psi_\hbar)_\hbar$ of quantum operators $(H_\hbar)_\hbar$, as $\hbar \to 0$.

Difficulty: behaviour of $(\psi_\hbar)_\hbar$ in $\mathcal{H}_\hbar$ is hard to capture

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Algebraic approach helpful

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Algebraic vector states $\omega_\hbar(\cdot) := \langle \psi_\hbar, (\cdot)\psi_\hbar \rangle$.

• Question: *Which set of physical observables makes the sequence $(\omega_\hbar)$ 'converge' as $\hbar \to 0$?*

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Strict deformation quantization

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Observables defined by quantization maps $Q_\hbar(f), (f \in C_0(X))$.

• Existence of **classical limit**, does

$$\omega_0(f) := \lim_{\hbar \to 0} \omega_\hbar(Q_\hbar(f)), \quad (f \in C_0(X)); \qquad (2)$$

exists as a **state** ω_0 on $A_0 = C_0(X)$ (i.e. a prob. measure on X)? Note: no subsequences involved!

• Spontaneous symmetry breaking (SSB): natural **emergent** phenomenon typically occuring in thermodynamic/classical limit. Difficulty: proving existence of SSB in such limits.

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Strict deformation quantization

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Existence of classical limit

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Rigorous notion of SSB in the classical limit:

pure ground states are not invariant, whilst invariant ground states are not pure.

• Consider collection of N two-level atoms corresponding to a spin chain of N sites described by a mean-field Hamiltonian H_N.

• Example: quantum Curie-Weiss spin Hamiltonian defined on $\mathcal{H}_N = \bigotimes_{n=1}^N \mathbb{C}^2$:

$$H_N \equiv H_N^{CW} = -\frac{J}{2N}\sum_{i,j=1}^N \sigma_3(i)\sigma_3(j) - B\sum_{i=1}^N \sigma_1(i), \tag{3}$$

with B magnetic field and J a coupling constant .

• Existence of a unique (up to phase) *permutation-invariant* ground state eigenvector $\Psi_N^{(0)}$ of H_N, i.e., $\Psi_N^{(0)} \in \mathcal{H}_N|_{\mathrm{Sym}(\mathbb{C}^N)} \cong \mathbb{C}^{N+1}$. The vector state

$$\omega_{1/N}^{(0)}(\cdot) = \langle \Psi_N^{(0)}, \cdot \Psi_N^{(0)} \rangle, \tag{4}$$

converges w.r.t. the observables:

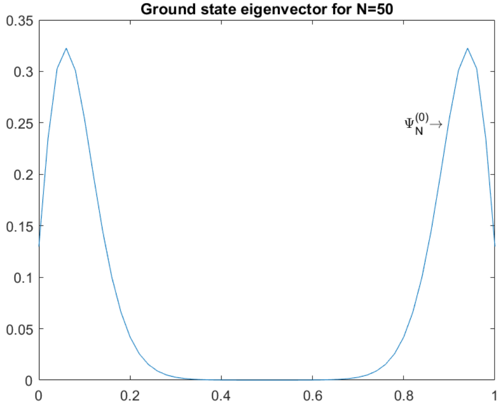
$$\omega_0^{(0)}(f) := \lim_{N\to\infty} \omega_{1/N}(Q'_{1/N}(f)) = \frac{1}{2}(f(\Omega_-) + f(\Omega_+)), \quad (f \in C(S^2)); \tag{5}$$

for some $\Omega_\pm \in (h^{CW})^{-1}(E) \subset S^2$ where $h^{CW}(x,y,z) = -\frac{1}{2}(x^2 + Bz)$, $((x,y,z) \in S^2)$, and E is the minima of h^{CW}. (Landsman,Moretti,v.d Ven, 2020).

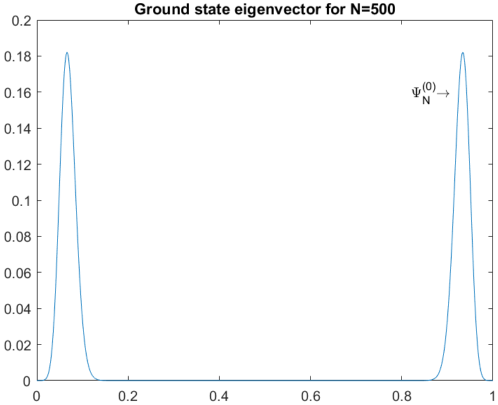
• The set $\{Q'_{1/N}(f) \mid f \in C(S^2)\} = M_{N+1}(\mathbb{C}) \to$ large set of 'physical observables' is considered.

• Proof Idea: **Localization** of eigenvectors $\Psi_N^{(0)}$ of H_N^{CW} $(N \to \infty)$ only depends on properties of h^{CW}.

• Existence of spontaneous symmetry breaking (SSB) in the classcial limit: pure ground states are not invariant, whilst invariant ground states are not pure $\longrightarrow$ emergence.



• The (pure) ground state eigenvector $\Psi_N^{(0)}$ of the quantum Curie-Weiss model is invariant under $\mathbb{Z}_2$- reflexion symmetry for any N.



• In the limit $N \to \infty$ the ground state eigenvector $\Psi_N^{(0)}$ 'decomposes' into two parts corresponding to the invariant (but not pure) state $\frac{1}{2}(f(\Omega_-)) + f(\Omega_+))$.

Generalizations: classical limit and SSB

Theorem (v.d.Ven, 2020)

If $(\lambda_N^{(i)})_N$ is a sequence of eigenvalues corresponding to a mean-field quantum spin Hamiltonian H_N such that $\lambda_N^{(i)}$ converges to some energy E^i, as $N \to \infty$. Then, if the corresponding sequence of eigenvectors $\Psi_N^{(i)}$ of H_N are permutation-invariant, they admit a classical limit in that $\exists n$

$$\lim_{N\to\infty} \langle \Psi_N^{(i)}, Q'_{1/N}(f)\Psi_N^{(i)} \rangle = \frac{1}{n}\sum_{j=1}^{n} f(\Omega_j), \quad (f \in C(S^2)); \tag{6}$$

where Ω_j are n distinct points in $(h^0)^{-1}(E^i)$ for a certain real valued-polynomial $h^0 : S^2 \to \mathbb{R}$.

• Consider $H_\hbar = -\hbar^2\Delta + V$, where $V : L^2(\mathbb{R}^n) \to \mathbb{R}$ is positive and blows up at infinity. Under the assumption that G acts continuously on $\mathbb{R}^{2n}$ as symplectomorphisms, leaves the set $h^{-1}(\min(h))$ invariant where $h(q,p) = p^2 + V(q)$, $((q,p) \in \mathbb{R}^{2n})$, and acts transitively on it, and the Hamiltonian $H_\hbar$ is G-invariant under the induced representation on $L^2(\mathbb{R}^n)$, we have the following result.

Theorem (Moretti, v.d.Ven, 2021)

If $(\psi_N^{(0)})_N$ is a sequence of eigenvectors of $H_\hbar$ corresponding to a non-degenerate minimal eigenvalues $\lambda_\hbar^{(0)}$ converging to $min(V)$ as $\hbar \to 0$. If G is compact, for any $\sigma_0 \in h^{-1}(min(h))$ one has

$$\lim_{\hbar\to 0} \langle \psi_\hbar^{(0)}, Q_{1/N}(f)\psi_\hbar^{(0)} \rangle = \int_G f(g\sigma_0)d\mu_g \quad (f \in C_0(\mathbb{R}^{2n})); \tag{7}$$

where μ_G is the normalized Haar measure of G.

• A rigorous way to understand the behaviour of eigenvectors of several opeators, e.g. Schrödiner operator with $SO(N)$-invariant potentials, in the semi-classical regime.

• Notion of SSB: if the classical limit exists, the limiting states breaks the symmetry if condition (1) holds and conidtions (2) or (3) are satisfied:

(1) It is a ground state in classical sense;
(2) It is a pure but not invariant state;
(3) It is an invariant, but a mixed state.

Research in progress

◇ Which other states admit a classical limit? (Think e.g. of pure (vector) states, Gibbs states, or β-KMS states describing thermal equilibrium at fixed temperature.)

◇ Generalize methods to more complicated many-body quantum systems and prove existence of SSB in the classical/thermodynamic limit, e.g. spin systems with nearest neighbor interactions, e.g. Heisenberg model.

◇ Exploit these techniques to detect phase transitions, by showing the presence of non-unique KMS states. Idea: 'classical' KMS states exist. Exploting quantization techniques would reduce the study of quantum KMS states in terms of their classical coutnerparts.

◇ Understanding BEC from a C^*-algebraic point of view.

◇ Not every classical theory is related to a underlying quantum theory. Only few pairs of a classical and a quantum C^*-algebra are known to connect in this way $\rightarrow$ open topic.

Thank you for your attention!

Algebraic ground states and SSB

Definition

Let A be a C^*-algebra with time evolution, i.e., a continuous homomorphism $\alpha : \mathbb{R} \to \text{Aut}(A)$. A *ground state* of (A, α) is a state ω on A such that:

1. ω is time independent, i.e., $\omega(\alpha_t(a)) = \omega(a)\ \forall a \in A\ \forall t \in \mathbb{R}$.
2. The generator h_ω of the ensuing continuous unitary representation

$$t \mapsto u_t = e^{ith_\omega} \tag{8}$$

of $\mathbb{R}$ on $\mathcal{H}_\omega$ has positive spectrum, i.e., $\sigma(h_\omega) \subset \mathbb{R}_+$, or equivalently $\langle \psi, h_\omega \psi \rangle \geq 0$ $(\psi \in D(h_\omega))$.

- The set of ground states forms a compact convex subset of $S(A)$, and we denote this set by $S_0(A)$. We moreover assume that pure ground states are pure states as well as ground states.

Definition

Suppose we have a C^*-algebra A, a time evolution α, a group G, and a homomorphism $\gamma : G \to \mathrm{Aut}(A)$, which is a symmetry of the dynamics α in that

$$\alpha_t \circ \gamma_g = \gamma_g \circ \alpha_t \quad (g \in G, t \in \mathbb{R}). \tag{9}$$

The G-symmetry is said to be spontaneously broken (at temperature $T = 0$) if

$$(\partial_e S_0(A))^G = \emptyset, \tag{10}$$

• Here $\mathscr{S}^G = \{\omega \in \mathscr{S} | \ \omega \circ \gamma_g = \omega \ \forall g \in G\}$, defined for any subset $\mathscr{S} \in S(A)$, is the set of G-invariant states in $\mathscr{S}$. (10) means that there are no G-invariant pure ground states. This means also that if spontaneous symmetry breaking occurs, then invariant ground states are not pure.