Bose-Einstein condensation for two-dimensional bosons in the Gross-Pitaevskii regime

Cristina Caraci

joint work with S. Cenatiempo and B. Schlein

University of Zurich

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Introduction

N interacting bosons in a 2D unit box $\Lambda = [-1/2, 1/2]^2$. The energy of the system is described by the Hamiltonian acting on $L^2_s(\Lambda^N)$

$$H_N = \underbrace{\sum_{j=1}^{N} -\Delta_{x_j}}_{\mathcal{O}(N)} + \underbrace{\sum_{i<j}^{N} e^{2N} V(e^N(x_i - x_j))}_{\mathcal{O}(N^2)}, \qquad (1)$$

Remarks:

- $V \in L^3(\mathbb{R}^2)$, spherically symmetric, compactly supported and pointwise non-negative (repulsive)
- The exponential scaling comes from the 2D scattering length $\mathfrak{a}$ of V

$$\frac{2\pi}{\log(R/\mathfrak{a})} = \inf_{\phi \in H^1(B_R)} \int_{B_R} \left[|\nabla \phi|^2 + \frac{1}{2} V |\phi|^2 \right] dx \qquad (2)$$

for $R > R_0$, with R_0 range of the potential and with $\phi(x) = 1$ for all x with $|x| = R$.

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The unique minimizer satisfies

$$-\Delta\phi^{(R)} + \frac{1}{2}V\phi^{(R)} = 0 \qquad (3)$$

For $R_0 < |x| \leq R$, we have

$$\phi^{(R)}(x) = \frac{\log(|x|/\mathfrak{a})}{\log(R/\mathfrak{a})}\,.$$

By scaling, $R = e^N R \rightsquigarrow \mathfrak{a}_N = e^{-N}\mathfrak{a}$, for the potential $e^{2N}V(e^N\cdot)$.

Bose-Einstein condensation

- **Aim:** Proof of **Bose-Einstein condensation** with optimal rate of convergence of the one-particle density matrix, i.e.

$$\gamma_N^{(1)}(x;y) = \int_{\Lambda^{N-1}} \overline{\psi_N(x, x_2, ..., x_N)} \psi_N(y, x_2, ..., x_N) dx_2 \cdots dx_N$$

for $\psi_N \in L^2_s(\Lambda^N)$, $\|\psi_N\| = 1$ or equivalently

$$\gamma_N^{(1)} = \mathrm{tr}_{2,\ldots,N} |\psi_N\rangle\langle\psi_N|$$

is such that

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq \frac{C}{N}$$

with $\varphi_0(x) = 1$ for all $x \in \Lambda$, the **condensate wave function**.

- **Meaning:** all particles up to a fraction vanishing in the limit $N \to \infty$, are condensated in the one-particle state φ_0.

Previous results 2D bosons in GP regime

- Lieb-Seiringer-Yngvason ('01,'06), Lieb-Seiringer ('01,'02) for bosons confined by external trapping potentials:
  - the ground state energy E_N of H_N is such that
  $$E_N = 2\pi N\big(1 + O(N^{-1/5})\big) \, ; \tag{4}$$
  - they imply BEC in the zero-momentum mode $\varphi_0(x) = 1$ for all $x \in \Lambda$, for any approximate ground state $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and
  $$\lim_{N\to\infty} \frac{1}{N}\langle \psi_N, H_N\psi_N\rangle = 2\pi, \tag{5}$$
  the one-particle reduced density matrix $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$ is such that
  $$1 - \langle \varphi_0, \gamma_N^{(1)}\varphi_0\rangle \le CN^{-\bar\delta} \tag{6}$$
  for a sufficiently small $\bar\delta > 0$.
- Schnee -Yngvason ('06) ground state energy at leading order and BEC for 3d bosons in a trap strongly confined in one direction

Our result

Theorem (C.-Cenatiempo-Schlein '20 arXiv:2011.05962)

Let $V \in L^3(\mathbb{R}^2)$ as before. Then there exists a constant $C > 0$ such that the ground state energy E_N of H_N satisfies

$$2\pi N - C \leq E_N \leq 2\pi N + C \log N \tag{7}$$

Furthermore, consider a sequence $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and such that

$$\langle \psi_N, H_N \psi_N \rangle \leq 2\pi N + K$$

for a $K > 0$. Then the reduced density matrix $\gamma_N^{(1)} = \mathrm{tr}_{2,\ldots,N} |\psi_N\rangle\langle\psi_N|$ associated with ψ_N is such that

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq \frac{C(1+K)}{N} \tag{8}$$

for all $N \in \mathbb{N}$ large enough.

Remarks & Strategy of the proof

Remarks:

- The condition $V \in L^3(\mathbb{R}^2)$ comes from properties of scattering equation;
- we have the **expected** order of the correction for the lower bound, but **logarithmic correction** for the upper bound;
- the convergence $1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq \frac{C(1+K)}{N}$ is expected to be **optimal**, but we need to choose $K = C \log N$ to hold $\langle \psi_N, H_N \psi_N \rangle \leq 2\pi N + K$.

Our **strategy:**

- line of the proof inspired by Boccato-Brennecke-Cenatiempo-Schlein ('20) for 3D bosons in Gross-Pitaevskii regime, **additional difficulties** to be overcome;
- Benedikter-de Oliveira-Schlein ('15) (Erdös-Schlein-Yau ('08)) to keep into account correlations between particles;
- ideas from Lewin-Nam-Serfaty-Solovej ('15) are used to factor out the condensate from the original Hamiltonian; localization techniques on the number of excitations (Lieb-Solovej '04);

Sketch of the proof

Idea: work in the Fock space and renormalize the Hamiltonian H_N to get the correct energy expectation. How?

$$e^{-A}e^{-B}U_N H_N U_N^* e^B e^A = \mathcal{R}_N = C_\mathcal{R} + Q_\mathcal{R} + \mathcal{C}_\mathcal{R} + \mathcal{H}_N + \mathcal{E}_R$$

To take into account correlation we use properties of the modified scattering equation

$$\Big(-\Delta + \frac{1}{2}V(x)\Big) f_\ell(x) = \lambda_\ell f_\ell(x) \qquad |x| \leq e^N \ell\,, \tag{9}$$

with $\ell = N^{-\alpha}$.

$$C_\mathcal{R} = \frac{1}{2}(N-1)\,\widehat{\omega}_N(0)(1-\mathcal{N}_+/N) + \frac{1}{2}\widehat{\omega}_N(0)\,\mathcal{N}_+\,(1-\mathcal{N}_+/N)$$

where $\omega_N \in L^\infty(\Lambda)$ **renormalized interaction potential** s.t. for any $p \in \Lambda_+^*$

$$\widehat{\omega}_N(p) := g_N\,\widehat{\chi}(p/N^\alpha)\,, \qquad g_N = 2N^{1-2\alpha}e^{2N}\lambda_\ell, \tag{10}$$

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$$\Big|\,\widehat{\omega}_N(0) - 4\pi\Big(1+\alpha\,\frac{\log N}{N}\Big)\,\Big| \leq \frac{C}{N}. \tag{11}$$

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