Irreversibility mitigation in unital non-Markovian dynamical maps

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Outline

Introduction: Physical context

- interest in the thermodynamics of quantum systems out-of-equilibrium
  - fundamental questions: emergence of thermodynamics from dynamics, thermalization vs localization, ...
  - applicative questions: quantum heat engines, energetically efficient quantum computations, ...
- non-Markovian dynamics necessary to describe quantum systems available in the lab
  - optical setups (Sci. Rep. 5, 17520 (2015))
  - N-V centers in diamonds (Phys. Rev. Lett. 121, 060401 (2018).)
  - ....
- Our work $\rightarrow$ intersection of the two (still almost unexplored)

Non-Markovian dynamical maps

- Finite dimensional Hilbert space $\mathcal{H} \simeq \mathbb{C}^n$
- Quantum evolution: one-parameter family of completely positive (CP) and trace preserving (TP) maps Λ_t
- Assume one can write $\Lambda_t = V_{t,s}\Lambda_s$, for any t, s such that $t \geq s \geq 0$ (always possible if Λ_s is invertible - in that case $V_{t,s} = \Lambda_t \Lambda_s^{-1}$)
- Different degrees of non-Markovianity (Chruscinski, Maniscalco, PRL 112, 120404 (2014))
  - Dynamics CP-divisible (Markovian): $V_{t,s}$ CP maps
  - Dynamics P-divisible but not CP-divisible (non-Markovian): $V_{t,s}$ P but not CP
  - Dynamics not P-divisible (essentially non-Markovian): $V_{t,s}$ not P for some t, s

Pauli channels - Kraus representation

- Class of unital dynamics ($\Lambda_t(\mathbb{1}) = \mathbb{1}$) for two-level systems

$$\Lambda_t(\varrho) = \sum_{\alpha=0}^{3} p_\alpha(t)\, \sigma_\alpha \varrho\, \sigma_\alpha, \tag{1}$$

  - $\{\sigma_\alpha\}_0^3 = \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ is the set of Pauli matrices plus the identity
  - the coefficients p_α obey the relation $\sum_\alpha p_\alpha(t) = 1$, $\forall t$ (trace preservation)
  - the initial condition $\Lambda_0 = \mathrm{id}$ implies that $p_0(0) = 1$ and $p_\alpha(0) = 0$ for $\alpha \neq 0$.
  - the map $\Lambda_t(\varrho)$ is CP if $p_\alpha(t) \geq 0$, $\forall t, \alpha$.

Pauli channels - Time dependent generator

- If Λ_t^{-1} exists, one can define a generator $\mathcal{L}_t = (\partial_t \Lambda_t)\Lambda_t^{-1}$.
  - invertibility: $p_1 + p_2$, $p_2 + p_3$ and $p_1 + p_3$ different from $1/2$
  - we restrict to the sector in parameter space containing the initial condition: $p_1 + p_2 < 1/2$, $p_2 + p_3 < 1/2$ and $p_1 + p_3 < 1/2$, $\forall t < \infty$
- Generator reads as follows (Chruscinski, Wudarski, PRA 91, 012104 (2015))

$$\mathcal{L}_t(\varrho) = \sum_{\alpha=1}^{3} \gamma_\alpha(t) \left(\sigma_\alpha \varrho \sigma_\alpha - \varrho\right), \tag{2}$$

  - where

$$\mathrm{e}^{-2\int_0^t \left(\gamma_\alpha(s)+\gamma_\beta(s)\right) ds} = 1 - 2(p_\alpha(t) + p_\beta(t)).$$

  - $\Lambda_t(\varrho)$ is CP-divisible iff $\gamma_\alpha(t) \geq 0 \ \forall t, \alpha$
  - $\Lambda_t(\varrho)$ is P-divisible iff $\gamma_\alpha(t) + \gamma_\beta(t) \geq 0$ with $\alpha, \beta = 1, 2, 3$ and $\alpha \neq \beta$

Stochastic entropy production

- Two-measurement protocol: $O^{\rm in} = \sum_k a_k^{\rm in} \Pi_k^{\rm in}$, $O^{\rm fin} = \sum_m a_m^{\rm fin} \Pi_m^{\rm fin}$
- For unital dynamics the sep reads (only for allowed transitions)

$$\Delta\sigma(a_m^{\rm fin}, a_k^{\rm in}) = \ln p(a_k^{\rm in}) - \ln p(a_m^{\rm fin}), \tag{3}$$

  where $p(a_m^{\rm fin})$ denotes the probability to measure the $m-$th outcome at the final time $t_{\rm fin}$.
- Probability distribution of $\Delta\sigma$:

$$\mathrm{Prob}(\Delta\sigma) = \sum_{k,m} \delta\left[\Delta\sigma - \Delta\sigma(a_m^{\rm fin}, a_k^{\rm in})\right] p(a_k^{\rm in}, a_m^{\rm fin}), \tag{4}$$

  where $\delta[\cdot]$ denotes the Dirac delta and $p(a_k^{\rm in}, a_m^{\rm fin}) = \mathrm{Tr}[\Pi_m^{\rm fin} \Lambda_{t_{\rm fin}}(\Pi_k^{\rm in})] p(a_k^{\rm in})$ with $p(a_k^{\rm in}) = \mathrm{Tr}[\varrho_0 \Pi_k^{\rm in}]$.

Irreversibility mitigation - part 1

- Objective: construct an example such that $\partial_t \langle \Delta\sigma \rangle < 0$ and $\partial_t \mathrm{Var}(\Delta\sigma) < 0$
- Derivative of the average (measure σ_3, initial state $|0\rangle\langle 0|$):

$$\partial_t \langle \Delta\sigma \rangle = (\gamma_1(t) + \gamma_2(t))\, \lambda_t \ln\left(\frac{1+\lambda_t}{1-\lambda_t}\right), \quad \lambda_t = \mathrm{e}^{-2\int_0^t (\gamma_1(s)+\gamma_2(s))\, ds}$$

  negative when $(\gamma_1(t) + \gamma_2(t)) < 0$ (no P-divisibility)
- Derivative of the variance

$$\partial_t \mathrm{Var}(\Delta\sigma) = 2 f_t\, \partial_t \langle \Delta\sigma \rangle\,,$$

  where the function f_t reads as follows

$$f_t = \frac{\lambda_t}{2} \ln\left(\frac{1+\lambda_t}{1-\lambda_t}\right) - 1\,.$$

Irreversibility mitigation - part 2

- Study the sign of f_t
  - Introduce $\phi_t \equiv \int_0^t(\gamma_1(s) + \gamma_2(s))\mathrm{d}s$
  - $f_t \geq 0$ iff $e^{-2\phi_t} \ln\left(\frac{1+e^{-2\phi_t}}{1-e^{-2\phi_t}}\right) \geq 2$
  - The function $x\ln(\frac{1+x}{1-x}) - 2$, with $0 \leq x < 1$, has an unique zero at $x^* \approx 0.834$ and is positive for $x \geq x^*$. Therefore the inequality is verified for
$$0 \leq \phi_t \leq \phi^* \equiv -\frac{1}{2}\ln(x^*) \approx 0.091$$
- $\partial_t \mathrm{Var}(\Delta\sigma) < 0$ and $\partial_t\langle\Delta\sigma\rangle < 0$ in time intervals where $\gamma(t) \equiv \gamma_1(t) + \gamma_2(t) < 0$ and $\phi_t < \phi^*$

Example

- Consider $\gamma(t) = \beta - \mathrm{e}^{-\alpha t}\left(1 - \mathrm{e}^{-\alpha t}\right), \quad \alpha, \beta > 0$
- Then $\phi_t = \beta t - \frac{\left(1-\mathrm{e}^{-\alpha t}\right)^2}{2\alpha}$
- Two zeros exist for γ at times t_1 and t_2 corresponding to

$$t_1 = -\frac{1}{\alpha}\ln\left(\frac{1}{2} + \frac{\sqrt{1-4\beta}}{2}\right), \quad t_2 = -\frac{1}{\alpha}\ln\left(\frac{1}{2} - \frac{\sqrt{1-4\beta}}{2}\right)$$

provided that $\beta < \frac{1}{4}$.

- $\gamma(t) < 0$ between t_1 and t_2 and positive otherwise
- (CP condition) $\beta \geq \frac{\left(1-\mathrm{e}^{-\alpha t}\right)^2}{2\alpha t}$, $\forall t > 0$. Therefore, one has to impose a lower bound $\overline{\beta}$ to β, which is given by

$$\overline{\beta} = \max_{t>0} \frac{\left(1-\mathrm{e}^{-\alpha t}\right)^2}{2\alpha t} \sim 0.2$$

Example

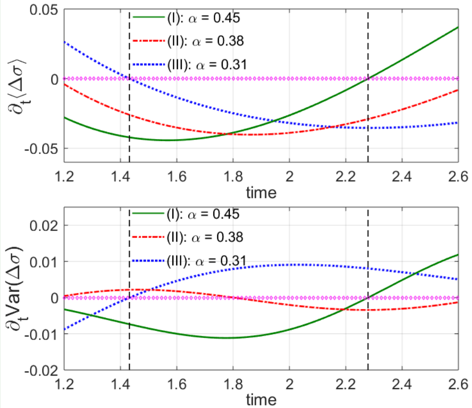

Figure: Time-derivatives of $\langle \Delta\sigma \rangle$ and $\mathrm{Var}(\Delta\sigma)$. We take $\beta = 0.23$ and three values of α, i.e. $0.31, 0.38, 0.45$ respectively represented by blue dotted, red dash-dotted and green solid lines.

Conclusion

- For essentially non-Markovian Pauli channels one can construct examples where both $\langle \Delta\sigma \rangle$ and $\mathrm{Var}(\Delta\sigma)$ are transiently decreasing

Outlook

- Non-unital qubit channels
- Concrete models of the environment producing such a phenomenology
- Higher dimension and possibly thermodynamic limit

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Thank you for your attention!