UNIVERSITÀ DEGLI STUDI DI BARI ALDO MORO

INFN Istituto Nazionale di Fisica Nucleare

Generic aspects of the resource theory of Quantum Coherence

Giovanni Gramegna
giovanni.gramegna@units.it

Joint work with: F.D. Cunden, P. Facchi, G. Florio

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Outline of the talk

- Resource Theoretic Approach
- Coherence Resource Theory
  - Free States
  - Allowed operations
- Deterministic state conversions
  - A criterion based on majorization relation
  - Majorization between random states as positivity of a random walk
  - Volume of convertible pairs for large systems
- Stochastic state conversions
  - Distribution of maximal success probability
  - Description in terms of a markovian random process

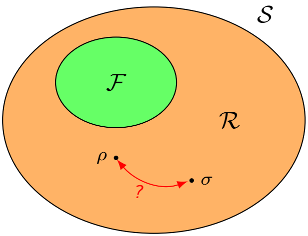
A quantum resource theory is defined by:

- **Allowed Operations:** a subset of the quantum operations satisfying some constraint, for example dictated by physical considerations
- **Free States:** states which can be obtained for free within the restriction imposed on the allowed operations

States which are not free represent a **resource**, since they can be used to perform operations which are not free

Resource convertibility.

- Some states are more useful than others
- Different resource states *may* be converted into each other using the allowed operations
- Not all resource states are comparable: there are pairs of states for which interconversion is not possible



Coherence Resource Theory: Free States

We denote the states over the Hilber space $\mathscr{H} = \mathbb{C}^n$ as:

$$\mathcal{S}_n = \{\rho : \mathbb{C}^n \to \mathbb{C}^n : \rho \geqslant 0, \operatorname{tr} \rho = 1\}$$

A particular base $\mathcal{B} = \{|i\rangle\}_{i=1}^n$ of $\mathscr{H}$ is selected as the **incoherent basis**.

Free states

States ρ which are diagonal in the "incoherent" basis $\{|i\rangle\}_{i=1}^n$:

$$\rho = \sum_{i=1}^{n} p_i \, |i\rangle\langle i| \, .$$

For **pure states** $\rho = |\psi\rangle\langle\psi| \, , |\psi\rangle \in \mathbb{C}^n, \langle\psi|\psi\rangle = 1$:

- Incoherent (free) states:

$$|\psi\rangle = |i\rangle \quad i = 1, \ldots, n$$

- Coherent (resource) states:

$$|\psi\rangle = \sum_{i=1}^{n} \psi_i \, |i\rangle$$

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Allowed operations (informally)

Transformations which do not generate coherent states starting from incoherent ones.

Examples:

- Permutations of the incoherent basis $U = \sum_{i=1}^{n} |\pi_i\rangle\langle i|$ are **allowed**:

$$|\psi\rangle = \sum_{i=1}^{n} \psi_i |i\rangle \longrightarrow |\psi'\rangle = U|\psi\rangle = \sum_{i=1}^{n} \psi_i |\pi_i\rangle = \sum_{i=1}^{n} \psi_{\pi_i^{-1}} |i\rangle \quad \checkmark$$

- Measurements in the incoherent basis $\{|i\rangle\}_{i=1}^{n}$ are **allowed**:

$$|\psi\rangle = \sum_{i=1}^{n} \psi_i |i\rangle \longrightarrow |i\rangle \text{ with probability } p_i = |\psi_i|^2 \quad \checkmark$$

- Generic unitary rotations are **not allowed**:

$$|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \qquad |2\rangle \rightarrow \frac{2}{\sqrt{2}}(|1\rangle - |2\rangle) \qquad X$$

For pure states, incoherent conversions are ruled by majorization relations. Given $|\psi\rangle = \sum_{i=1}^{n} \psi_j |i\rangle$, we introduce the "coherence vector":

$$\mu(\psi) := (|\psi_1|^2, \ldots, |\psi_n|^2)$$

Theorem: Deterministic conversion among pure states

A pure state $|\psi\rangle$ can be transformed into $|\psi'\rangle$ under IO if and only if $\mu(\psi) \prec \mu(\psi')$

Majorization

For $x, y \in \mathbb{R}^n$, we write $x \prec y$ if and only if:

$$\sum_{j=1}^{k} x_j^{\downarrow} \leqslant \sum_{j=1}^{k} y_j^{\downarrow} \qquad k = 1, \ldots, n$$

with equality for $k = n$ and where $x_j^{\downarrow}$ denotes the components of x arranged in decreasing order, i.e. $x_1^{\downarrow} \geqslant \cdots \geqslant x_n^{\downarrow}$.

Example

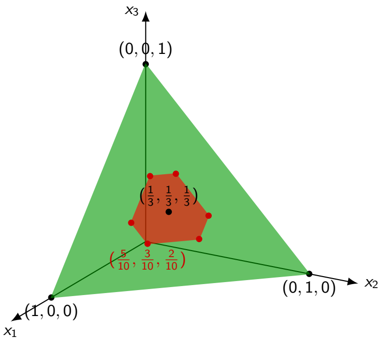
$\mathscr{H} = \mathbb{C}^3$,

$$|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \ |2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, \ |3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$$

$$|\varphi\rangle = \sqrt{\frac{5}{10}}|1\rangle + \sqrt{\frac{3}{10}}|2\rangle + \sqrt{\frac{2}{10}}|3\rangle$$

$$|\chi\rangle = \sqrt{\frac{9}{20}}|1\rangle + \sqrt{\frac{9}{20}}|2\rangle + \sqrt{\frac{2}{20}}|3\rangle$$



We easily see:

- $\mu(\psi) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \prec (\frac{5}{10}, \frac{3}{10}, \frac{2}{10}) = \mu(\varphi) \Leftrightarrow |\psi\rangle \to |\varphi\rangle$
  $\mu(\psi) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \prec (\frac{9}{20}, \frac{9}{20}, \frac{2}{20}) = \mu(\chi) \Leftrightarrow |\psi\rangle \to |\chi\rangle$
- $\mu(\varphi) \not\prec \mu(\chi)$, $\mu(\chi) \not\prec \mu(\varphi)$: the pair of states $|\varphi\rangle$ and $|\chi\rangle$ cannot be deterministically converted into each other.

Majorization as positivity of a random walk

Conjecture

Let $|\psi\rangle, |\psi'\rangle \in \mathbb{C}^n$ be random uniform (Haar) states. Then:

$$P(|\psi\rangle \to |\psi'\rangle) = P(\mu(\psi) \prec \mu(\psi')) \to 0 \text{ as } n \to \infty$$

For $\mu, \mu' \in \Delta_{n-1}$, define, for $k = 1, \dots, n$:

$$\delta_k = \mu'^{\downarrow}_k - \mu^{\downarrow}_k$$
$$S_k = S_{k-1} + \delta_k, \quad S_0 = 0,$$

Then:

$$P(\mu \prec \mu') = P\left(\sum_{j=1}^{k} \delta_j \geq 0, \text{ for } 1 \leq k \leq n\right) = P((S_k) \text{ stays positive}).$$

(S_k) is *not* a simple random walk:

- δ_k are not i.i.d. (the components of $x^{\downarrow}_k$ and $y^{\downarrow}_k$ are neither independent nor identically distributed);
- The normalization of x and y implies that (S_k) is a ‘random bridge’ starting at $S_0 = 0$ and ending at $S_n = 0$.

Deterministic conversion among random pure states

Theorem: Most pairs are not IO-interconvertible

Let $|\psi\rangle$ and $|\psi'\rangle$ be independent random (Haar uniform) pure states on $\mathbb{C}^n$. Then

$$\lim_{n\to\infty} P(|\psi\rangle \to |\psi'\rangle) = 0$$

Sketch of the proof:

- For a random (Haar uniform) state $|\psi\rangle \in \mathbb{C}^n$, the coherence vector $\mu(\psi) = (|\psi_1|^2, \ldots, |\psi_n|^2)$ is a uniform point on the simplex
- $|\psi\rangle \to |\psi'\rangle \Leftrightarrow \mu \prec \mu'$
- For $1 \leqslant k \leqslant n$, define:

$$\pi_{n,k} := P\left(\sum_{i=n-j+1}^{n} \mu_i^{\downarrow} \geq \sum_{i=n-j+1}^{n} \mu_i'^{\downarrow}, \text{ for all } 1 \leq j \leq k \right)$$

$$\pi_{n,n} = P\left(\mu \prec \mu'\right), \quad \text{and} \quad \pi_{n,n} \leq \pi_{n,k}.$$

- The idea then is to prove:

$$\lim_{k\to\infty} \left(\lim_{n\to\infty} \pi_{n,k} \right) = 0$$

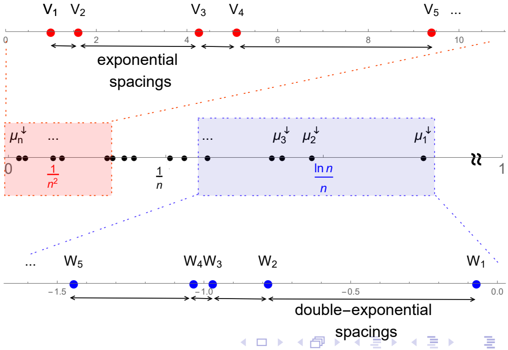
Lemma: Smallest components of a uniform point on the simplex

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be a uniform point in Δ_{n-1}. Then, the rescaled vector of the k smallest components $\left(n^2 \mu^{\downarrow}_{n-j+1}\right)_{1\leq j\leq k}$ converges in distribution to a Markovian process $(V_1, V_2, \ldots, V_k)$, with initial and transition densities

$$f_{V_1}(v) = \exp(-v) 1_{v\geq 0},$$
$$f_{V_{j+1}|V_j}(u|v) = \exp(v-u) 1_{u\geq v};$$

$$V_j = \sum_{i=1}^{j} X_i$$

$$X_i \sim \mathrm{Exp}(1)$$



Going back to the proof...

$$\pi_{n,k} := P\left(\sum_{i=n-j+1}^{n} \mu_i^{\downarrow} \geq \sum_{i=n-j+1}^{n} \mu_i'^{\downarrow}, \text{ for all } 1 \leq j \leq k \right)$$

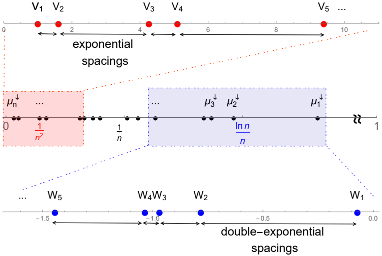
We want to prove:

$$\lim_{k\to\infty} \left(\lim_{n\to\infty} \pi_{n,k} \right) = 0$$

$$p_k := \lim_{n\to\infty} \pi_{n,k}$$

$$= P\left(\sum_{i=1}^{j} V_i \geq \sum_{i=1}^{j} V_i', \text{ for all } 1 \leq j \leq k \right)$$

$$= P\left(I_j \geq 0, \text{ for all } 1 \leq j \leq k \right)$$



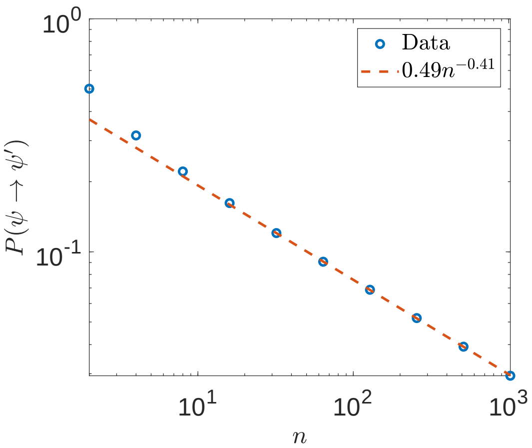
where the process $(I_j)_{j\in\mathbb{N}}$ is an *Integrated Random Walk*:

$$I_j = \sum_{i=1}^{j} \widetilde{V}_i, \qquad \widetilde{V}_j = V_j - V_j' = \sum_{i=1}^{j} (X_i - X_i') = \sum_{i=1}^{j} \widetilde{X}_i$$

Then $p_k \to 0$ follows from standard arguments of probability theory.

Numerically, we also find the persitence exponent, i.e. θ such that

$$P(\mu \prec \mu') \sim bn^{-\theta}$$



Stochastic conversions among pure states

For pure states $|\psi\rangle = \sum_{i=1}^{n} \psi_i |i\rangle$ stochastic conversions are also ruled out by the coherence vector

$$\mu(\psi) := (|\psi_1|^2, \ldots, |\psi_n|^2)$$

Theorem: Maximal success probability for a stochastic conversion

The maximal conversion probability under IO from $|\psi\rangle$ to $|\psi'\rangle$ is given by

$$\Pi(\mu(\psi), \mu(\psi')) = \min_{1 \leqslant k \leqslant n} \frac{\sum_{j=k}^{n} \mu(\psi)_j^{\downarrow}}{\sum_{j=k}^{n} \mu(\psi')_j^{\downarrow}}$$

In particular:

$$\Pi(\mu(\psi), \mu(\varphi)) = 1 \Leftrightarrow \mu(\psi) \prec \mu(\varphi)$$

and a conclusive conversion $|\psi\rangle \to |\varphi\rangle$ is possible.
From the previous example:

$$\mu(\varphi) = \left(\frac{5}{10}, \frac{3}{10}, \frac{2}{10}\right), \qquad \mu(\chi) = \left(\frac{9}{20}, \frac{9}{20}, \frac{2}{20}\right)$$

$|\varphi\rangle \nrightarrow |\chi\rangle$, nor $|\chi\rangle \nrightarrow |\varphi\rangle$ but $\Pi(\varphi \to \chi) = 8/9 > 0$ and $\Pi(\chi \to \varphi) = 9/10 > 0$

Distribution of the maximal success probability of IO-conversion

We investigated numerically the distribution of the maximal success probability of IO-conversion, i.e. the function

$$F_n(p) := P(\Pi(\mu(\psi), \mu(\psi')) \leqslant p)$$

for $|\psi\rangle, |\psi'\rangle$ sampled uniformly and independently from the unit sphere in $\mathbb{C}^n$. Numerical evidence shows convergence towards a limit distribution:

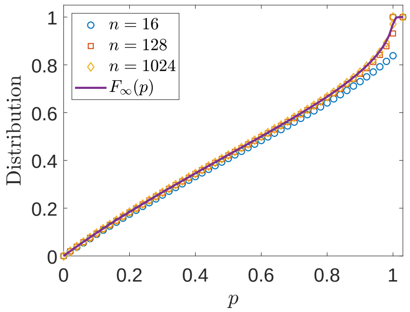
$$\lim_{n\to\infty} F_n(p) = F_\infty(p)$$

$$F_\infty(p) := P(\Pi^\infty(V, V') \leqslant p)$$

$$\Pi^\infty(V, V') := \inf_{k\geqslant 1} \frac{\sum_{j=1}^k V_j}{\sum_{j=1}^k V_j'}$$

$$V_j = \sum_{i=1}^j X_i, \quad V_j' = \sum_{i=1}^j X_i'$$

$$X_i, X_i' \sim \mathrm{Exp}(1)$$



Conclusions

- Incoherent conversions between pure states $|\psi\rangle = \sum_{i=1}^{n} \psi_i |i\rangle$ are determined by the coherence vector $\mu(\psi) = (|\psi_1|^2, \dots, |\psi_n|^2)$
  - Deterministic conversions are ruled out by a majorization relation

$$\psi \to \psi' \quad \Leftrightarrow \quad \mu(\psi) \prec \mu(\psi')$$

  - Stochastic conversions are determined by a function $\Pi(\mu(\psi), \mu(\psi'))$ whose asymptotic distribution for random pairs is determined only by the smallest components of $\mu(\psi), \mu(\psi')$
- We proved most pairs of random pure states are non-convertible as $n \to \infty$:

$$P(\psi \to \psi') \to 0,$$

which can be also expressed as a result for majorization between random pairs μ, μ' on the probability simplex:

$$P(\mu \prec \mu') \to 0 \quad as \quad n \to \infty$$

Some relevant references

Main results of the talk:

- F. D. Cunden, P. Facchi, G. Florio, and G. Gramegna, "Generic aspects of the resource theory of quantum coherence", Phys. Rev. A **103**, 022401 (2021)

For simple introductions on Quantum Resource Theories:

- G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens, N. Y. Halpern, "The resource theory of informational nonequilibrium in thermodynamics", Physics Reports, **583**, 1-58 (2015)
- E. Chitambar, G. Gour, "Quantum Resource Theories", Rev. Mod. Phys. **91**, 025001 (2019)

Related results in the Resource Theory of Entanglement:

- F. D. Cunden, P. Facchi, G. Florio, and G. Gramegna, "Volume of the set of LOCC-convertible quantum states", J. Phys. A **53**, 175303 (2020)

Thanks for the attention

The incoherent operations (IO) on the Hilbert space $\mathscr{H}$, with the incoherent basis $\{|i\rangle\}_{i=1}^{n}$, are defined as maps

$$\mathcal{E} : \mathcal{S}(\mathscr{H}) \to \mathcal{S}(\mathscr{H})$$

$$\rho \mapsto \mathcal{E}(\rho) = \sum_{\alpha=1}^{n} K_\alpha \rho K_\alpha^\dagger$$

where the Kraus operators $K_\alpha : \mathscr{H} \to \mathscr{H}$ are in the form

$$K_\alpha = \sum_{i=1}^{n} c_{\alpha,i} \, |f_\alpha(i)\rangle\langle i| \, ,$$

for some $c_{\alpha,i} \in \mathbb{C}$ and $f_\alpha : \{1, \dots, n\} \to \{1, \dots, n\}$.