Excitation spectrum of Bose Gases beyond the Gross-Pitaevskii limit

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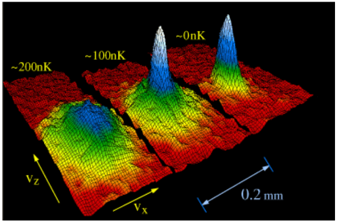
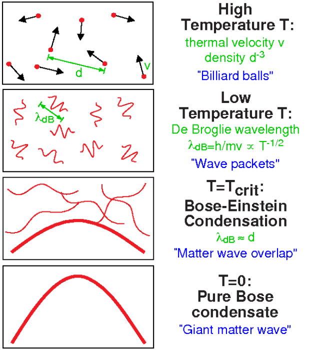
Bose-Einstein condensate: Macroscopic occupation of one-particle state

- Predicted by Bose, Einstein for non-interacting systems in the ‘20s;
- Accessible to experiments since the ‘90s.





N bosons in a fixed box $\Lambda = [-1/2, 1/2]^3$, $N \to \infty$. The hamiltonian of the system is given by:

$$H = -\sum_{i=1}^{N} \Delta_i + \sum_{1 \le i < j \le N} V_N(x_i - x_j) \qquad \text{acting on } L^2_s(\Lambda^N),$$

V_N = rescaled version of V: repulsive, radial, short-range.

- Thermodynamic limit: $V_N(x) = N^{2/3} V(N^{1/3} x)$;
- Gross-Pitaevskii scaling: $V_N(x) = N^2 V(Nx)$.
  Range of interaction: $N^{-1} \ll \rho^{-1/3} = N^{-1/3} \to$ **dilute**;
- Intermediate scaling: $V_N(x) = N^{2-2\kappa} V(N^{1-\kappa} x)$ for $\kappa \in (0, 2/3)$.

Previous results in GP (non-exhaustive)

Ground-state energy. [Lieb-Yngvason '98], [Lieb-Seiringer-Yngvason '00], [Nam-Rougerie-Seiringer '16]:

$$E_N = 4\pi\mathfrak{a}_0 N + o(N).$$

BEC. [Lieb-Seiringer '02, '06], [NRS '16] showed for low-energy states ψ_N

$$\lim_{N\to\infty} \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle = 1, \qquad \text{where } \gamma_N^{(1)} = \mathrm{Tr}_{2,\dots,N} |\psi_N\rangle\langle\psi_N|,$$

and $\varphi_0 =$ minimizer of GP functional.

Optimal rate of BEC. [Boccato-Brennecke-Cenatiempo-Schlein '20], [Nam-Napiorkowski-Ricaud-Triay '20], [Brennecke-Schlein-Schraven '21]:

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \le C N^{-1}.$$

Low-lying excitation spectrum. [BBCS '19] using optimal rate of BEC.

Beyond GP

Ground-state energy. [Yin-Yau '09], [Fournais-Solovej '20], [Basti-Cenatiempo-Schlein '21]: second-order expansion in the thermodynamic limit (Lee-Huang-Yang formula)

$$e(\rho) = 4\pi\mathfrak{a}\rho^3\left[1+\frac{128}{15\sqrt{\pi}}(\rho\mathfrak{a}^3)^{1/2}+o((\rho\mathfrak{a}^3)^{1/2})\right].$$

BEC. Follows from [Lieb-Seiringer '02] for $\kappa < 1/10$.

[Adikhari-Brennecke-Schlein '20]: for $\kappa < 1/44$, better rate for κ small and bounds of the form:

$$\langle \xi_N, \mathcal{H}_N \mathcal{N}_+^{(j-1)} \xi_N \rangle \lesssim N^{(44\kappa+\varepsilon)j}.$$

[Fournais '20]: condensation for $\kappa < 2/5$.

Theorem [Brennecke, C., Schlein; in preparation]: *For $\kappa, \mu > 0$ small enough and $V \in L^3(\mathbb{R}^3)$, there exists $\varepsilon > 0$ such that*

$$\begin{aligned} E_N =& 4\pi \mathfrak{a}_0 N^\kappa (N-1) + e_\Lambda (\mathfrak{a}_0 N^\kappa)^2 + \mathcal{O}(N^{-\varepsilon}) \\ &+ \frac{1}{2} \sum_{p \in 2\pi\mathbb{Z}^3_+} \left[\sqrt{|p|^4 + 16\pi \mathfrak{a}_0 N^\kappa |p|^2} - |p|^2 - 8\pi \mathfrak{a}_0 N^\kappa + \frac{(8\pi \mathfrak{a}_0 N^\kappa)^2}{2|p|^2} \right], \end{aligned}$$

with

$$e_\Lambda = 2 - \lim_{M \to \infty} \sum_{\substack{p \in \mathbb{Z}^3_+ \\ |p_1|, |p_2|, |p_3| \leq M}} \frac{4 \cos(|p|)}{|p|^2}.$$

Moreover, the spectrum of $H_N - E_N$ below the threshold $N^{\kappa/2+\mu}$ is given by

$$\sum_{p \in 2\pi\mathbb{Z}^3_+} n_p \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 N^\kappa |p|^2} + \mathcal{O}(N^{-\varepsilon}),$$

where $n_p \in \mathbb{N}$ for all $p \in 2\pi\mathbb{Z}^3_+$.

Remarks:

- Extends the results of [BBCS '19] in GP to $\kappa > 0$;
- Second-order correction consistent with Lee-Huang-Yang formula:

$$E_N = 4\pi\mathfrak{a}_0 N^{1+\kappa} + 4\pi\frac{128}{15\sqrt{\pi}}(\mathfrak{a}_0 N^\kappa)^{5/2} + \mathcal{O}(N^{2\kappa});$$

- Dispersion relation linear for low momenta $\epsilon(p) \sim |p|$ used to explain emergence of superfluidity.

Thanks for your attention!