Bose-Einstein condensation with Optimal Rate for Trapped Bosons in the Gross-Pitaevskii Regime

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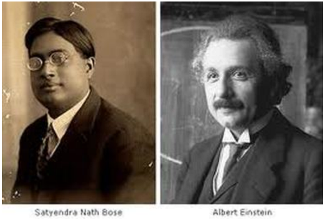
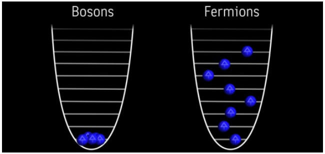
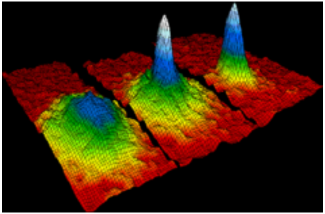
joint work with C. Brennecke and B. Schlein.

Bose-Einstein condensate



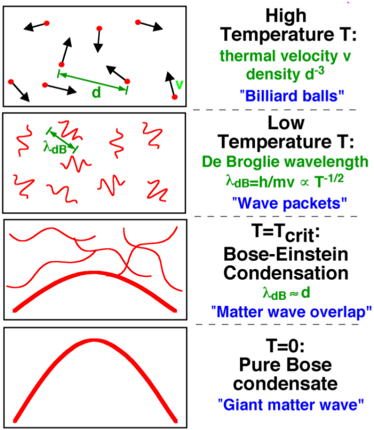
Symmetric wavefunction





Velocity-distribution rubidium

Bose-Einstein condensation: Heuristics



Reduced one-particle density matrix

Let $\psi_N \in L^2_s(\mathbb{R}^{3N})$ be a normalized symmetric (i.e. bosonic) wavefunctions. We define the **reduced one-particle density** of ψ_N to be

$$\gamma_N^{(1)}(x,y) := \int_{\mathbb{R}^{3(N-1)}} \psi_N(x,X)\overline{\psi_N(y,X)}dx_2\dots dx_N.$$

For an observable $\mathcal{O}$ that only depends on 1-particle, i.e. is of the form

$$\mathcal{O} = \mathrm{Sym}_N(\mathcal{O}^{(1)} \otimes Id^{\otimes(N-1)}),$$

then

$$\langle \psi_N, \mathcal{O}\psi_N\rangle = Ntr(\mathcal{O}^{(1)}\gamma_N^{(1)}).$$

For example, the kinetic energy $\mathcal{K} = \frac{\hbar^2}{2m}\sum_{j=1}^N -\Delta_{x_j}$ is a one-particle observable

However, the interaction energy $\mathcal{V}_N = \sum_{1\leq i<j\leq N} V_N(x_i - x_j)$ would be a $2-$particle observable.

Definition complete asymptotic B.E.C.

Theorem 1

Let $(\gamma_N^{(1)})_{N\geq 1}$ be a the sequence of one-particle reduced density matrices corresponding to the sequence $(\gamma_N)_{N\geq 1}$ of N-body bosonic states on the Hilbert space $L^2(\Omega)^{\otimes N}$. Let $\varphi \in L^2(\Omega)$ with $\|\varphi\|_2 = 1$. Then the following are equivalent:

1. $\gamma_N^{(1)} \overset{N\to\infty}{\longrightarrow} |\varphi\rangle\langle\varphi|$ *weakly-$*$ in trace class*
2. $\gamma_N^{(1)} \overset{N\to\infty}{\longrightarrow} |\varphi\rangle\langle\varphi|$ *in the Hilbert-Schmidt norm*
3. $\gamma_N^{(1)} \overset{N\to\infty}{\longrightarrow} |\varphi\rangle\langle\varphi|$ *in the trace norm*
4. $\langle\varphi, \gamma_N^{(1)}\varphi\rangle \overset{N\to\infty}{\longrightarrow} 1.$

We say that $(\gamma_N^{(1)})_{N\geq 1}$ exhibits **complete asymptotic B.E.C** in the condensate wave function $\varphi \in L^2(\Omega)$ iff any of the equivalent statements in the above theorem hold.

Gross-Pitaevskii regime

We consider a system of N-particles with Hamiltonian

$$H_N = \sum_{j=1}^{N} \left[-\Delta_{x_j} + V_{ext}(x_j)\right] + \sum_{1\leq i<j\leq N} N^2 V(N(x_i - x_j))$$

- Here V_{ext} is a trapping potential, i.e. $\lim_{R\to\infty} \sup_{|x|\leq R} V_{ext}(x) = \infty$ and spherically symmetric.
- Repulsive, radially symmetric and short-range interaction, i.e. $V \geq 0$ is spherically symmetric with compact support.

More technical assumptions

- $V_{ext} \in C^3(\mathbb{R}^3; \mathbb{R})$
- $\exists\, C > 0\ \forall\, x, y \in \mathbb{R}^3 : V_{ext}(x+y) \leq C(V_{ext}(x) + C)(V_{ext}(y) + C)$, $\nabla V_{ext}, \Delta V_{ext}$ have at most exponential growth as $|x| \to \infty$.

We also have the **Gross-Pitaevskii functional**

$$\mathcal{E}_{GP}(\psi) = \int_{\mathbb{R}^3} \left(|\nabla\psi(x)|^2 + V_{ext}(x)|\psi(x)|^2 + 4\pi\mathfrak{a}_0|\psi(x)|^4\right) dx.$$

Theorem 2 (Brennecke, Schlein, Schraven '21)

Let V and V_{ext} be "sufficiently nice" and let E_N denote the ground state energy of H_N. Then, there exists a constant $C > 0$ such that

$$E_N \geq N\mathcal{E}_{GP}(\varphi) - C \tag{1}$$

and

$$H_N \geq N\mathcal{E}_{GP}(\varphi) + C^{-1} \sum_{i=1}^{N} \big(1 - |\varphi\rangle\langle\varphi|_i\big) - C.$$

In particular, if $\psi_N \in L^2_s(\mathbb{R}^{3N})$ with $\|\psi_N\| = 1$ is an sequence of approximate ground states such that

$$\langle \psi_N, H_N \psi_N \rangle \leq N\mathcal{E}_{GP}(\varphi) + \zeta,$$

for a $\zeta > 0$, then the reduced density $\gamma_N^{(1)}$ associated with ψ_N satisfies

$$1 - \langle \varphi, \gamma_N^{(1)} \varphi \rangle \leq \frac{(C + \zeta)}{N}.$$

Previous results

Our result extents the following previous results:

- Lieb, Seiringer '02: Show convergence, but without optimal rate (both in the box and in the trap).
- Boccato, Brennecke, Cenatiempo, Schlein '20: Optimal rate in the box.
- Nam, Napiórkowski, Ricaud, Triay '20: Optimal rate in both the box and the trap. Need to impose smallness condition for the scattering length for the lower bound.