

Applications to the dynamics of Open Quantum Systems:

The dynamics of a small system of interest, weakly coupled to an environment can be described in the Markovian approximation by an effective lin. evol. equation:

$$\begin{cases} \dot{\rho}(t) = L(\rho(t)), \quad t \geq 0, \text{ where } \rho(t) \in \mathcal{T}(\mathcal{H}) \\ \rho(0) = \rho_0 \end{cases}$$

is a density matrix

$L: \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ the Lindbladian

Actually, requirements from Physics are $\forall t \geq 0$,

- $\rho(t) = e^{tL}(\rho_0) \geq 0$, $\mathrm{tr} \rho(t) = 1$, e^{tL} is a CP (completely positive) map.

$(T \in \mathcal{L}(\mathcal{L}(\mathcal{H})) \text{ CP : } \forall n \in \mathbb{N}^*, T \otimes \mathbb{I}_{id_n} \in \mathcal{L}(\mathcal{L}(\mathcal{H}) \otimes M_n(\mathbb{C})) \text{ is positive})$

Facts: • $(e^{tL})_{t \geq 0}$ is a norm-continuous family of CPTP maps on $\mathcal{L}(\mathcal{H})$

$$\Leftrightarrow L(\rho) = -i[H, \rho] + \sum_{j \in J} V_j \rho V_j^* - \frac{1}{2} \{V_j^* V_j, \rho\}, \quad \forall \rho \in \mathcal{T}(\mathcal{H})$$

where $H = H^* \in \mathcal{L}(\mathcal{H})$, $V_j \in \mathcal{L}(\mathcal{H})$, $\forall j \in J$, $\sum_{j \in J} V_j^* V_j$ is ^{shang CV} bdd.

Gorini, Kossakowski, Sudarshan '76 (matrix case), Lindblad '76 (bounded case).

- H and $\{V_j\}_{j \in J}$ are not unique.

- e^{tL} positive $\Rightarrow \|e^{tL}\|_{op} = 1, \forall t \geq 0$, (contraction semigroup)

as a map on $\mathcal{T}(\mathcal{H})$.

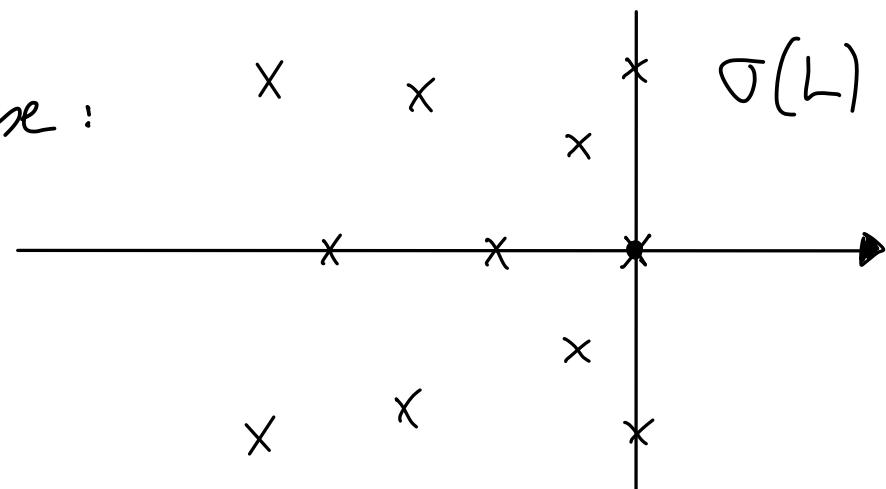
- Matrix case $L(\mathbb{C}^d)$ is a Hilbert space with $(A, B) = \text{tr}(A^*B)$ as scal. prod.

$L \neq L^*$, not normal. $0 \in \sigma(L)$; $\sigma(L) \subset \{\operatorname{Re} z \leq 0\}$.

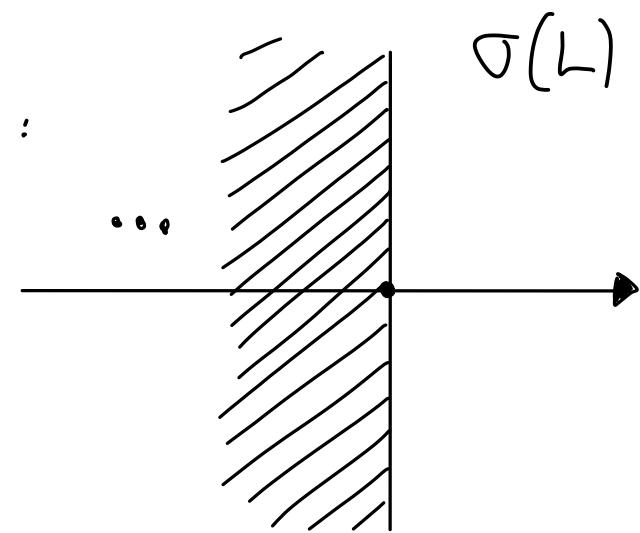
$\lambda \in \sigma(L)$ with $\operatorname{Re} \lambda = 0 \Rightarrow$ no associated eigen-mul. potent

$$L(f^*) = (L(f))^* \Rightarrow \sigma(L) = \overline{\sigma(L)}$$

Matrix case:



In general:



- When $H(t)$, $V_j(t)$, $j \in J$ are time dependent, L_t becomes the generator of a two-parameter evolution operator $U(t, t_0)$ on $\mathcal{T}(\mathcal{H})$, if $t \mapsto L_t$ shr-C°.

$$\forall t \geq t_0, \quad \partial_t U(t, t_0) f = L_t(U(t, t_0) f), \quad \forall f \in \mathcal{T}(\mathcal{H}); \quad U(t_0, t_0) = \mathbb{I},$$

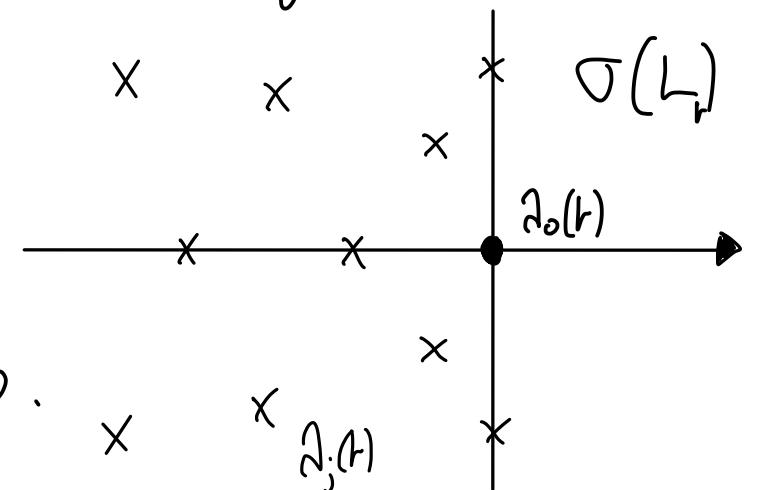
o.b. $\|U(t, t_0) f\|_{op} \leq \|f\|_1, \quad \forall f \in \mathcal{T}(\mathcal{H}).$

- A diabatic evolution: $\varepsilon \dot{f} = L_t(f)$; $t \in [0,1]$, assuming $t \mapsto L_t$ on C^2 .

- Matrix case: $\mathcal{H} = \mathbb{C}^d$

Assume $\sigma(L_t) = \{\lambda_0(t), \lambda_1(t), \dots, \lambda_{d^2-1}(t)\}$ o.b. $\text{mult.}(\lambda_j(t)) = m_j$, $\forall t \in [0,1]$.

and $L_t = \sum_{j=0}^N \lambda_j(t) P_j(t) \in \mathcal{L}(M_d(\mathbb{C}))$ where $N \leq d^2-1$,



i.e. no Jordan blocks. $\lambda_0(t) \equiv 0$, $\operatorname{Re} \lambda_j(t) \leq 0$.

Then, $f_\varepsilon(t)$ sol. to $\varepsilon \dot{f}_\varepsilon(t) = L_t(f_\varepsilon(t))$, $f_\varepsilon(0) = f = P_0^{(0)} f \in M_d(\mathbb{C})$ satisfies

$$f_\varepsilon(t) = W(t) P_0^{(0)} f + O(\varepsilon), \quad \text{unif in } t \in [0,1], \text{ where}$$

$W(t)$ defined by $W'(t) = \left(\sum_{j=0}^N P_j'(t) P_j(t) \right) W(t)$; $W(0) = \mathbb{I}$ o.b.

$$W(t) P_j^{(0)} = P_j(t) W(t), \quad \forall t \in [0,1], \forall j \in \{0, \dots, N\}.$$

Proof:

i) the intertwining property of ω is similar to the case considered so far.

ii) $\sigma(iL_r) = \{i\lambda_j(r)\}_{j=0}^N$ with $\operatorname{Im}(i\lambda_j(r)) \leq 0$.

iii) $\phi_\varepsilon(t)$ s.t. $i\phi'_\varepsilon(t) = (\bar{W}(H)iL_r W(t))\phi_\varepsilon(t)$; $\phi_\varepsilon(0) = 1$

$$\phi_\varepsilon(t) = \sum_{j=0}^N P_j(0) e^{\frac{1}{\varepsilon} \int_0^t \lambda_j(s) ds} = P_0(0) + \sum_{j>0} P_j(0) e^{\frac{1}{\varepsilon} \int_0^t \lambda_j(s) ds}.$$

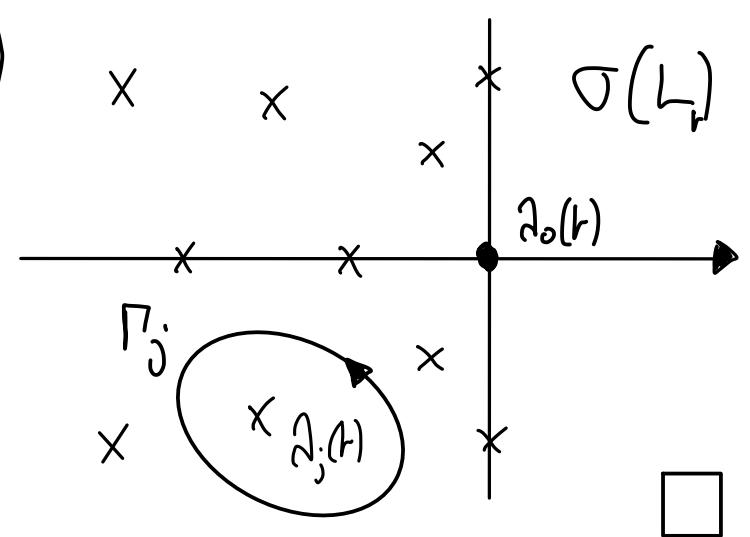
s.t. $\sup_{t \in [0,1]} \sup_{\varepsilon > 0} \|\phi_\varepsilon(t)\| < C_1 \Rightarrow \sup_{t \in [0,1]} \sup_{\varepsilon > 0} \|V_\varepsilon(t)\| \leq C_2$

(b) Integration by parts on $\sum_j P'_j S_j$ in place of $[P', P]$ using

$$R_j(B) := -\frac{1}{2i\pi} \int_{\Gamma_j} (H-z)^{-1} B (H-z)^{-1} dz \quad (\text{with } H=iL)$$

$$P'_j S_j = [P'_j S_j, S_j] = [H, R_j(P'_j S_j)]$$

v) $U_\varepsilon(t) = V_\varepsilon(t) + O(\varepsilon)$.



□

Remarks:

- Generically, $\text{mult}(\mathcal{J}(t)) = 1$; In any case, no nilpotent associated to $\mathcal{J}(t) \equiv 0$.
- In case nilpotents are present, see J. '08 (exponentially accurate)
- For adiabatic evolution for generators of contractions in Banach spaces,
see Aron Fraas Graf Grech '12, J. '08 (with and without gap).
- Relevant references (past and present):
Nenciu-Rasche '92, Abn-Salem, Fröhlich '05-'06, Sarandi, Lidor '05
Aron Fraas Graf Grech '11, Aron Fraas Graf '12, Fraas, Hänggli '17,
- Nice application of the matrix case to dephasing Lindbladians, see below.

- Dephasing Lindbladians:

Auron et al '12

$\mathcal{H} = \mathbb{C}^d$, $\mathcal{L}(\mathbb{C}^d)$ Hilbert with $(A, B) = \text{Tr}(A^* B)$.

$$L(\rho) = -i[H, \rho] + \sum_{j \in J} V_j \rho V_j^* - \frac{1}{2} \{ V_j^* V_j, \rho \} ; \quad J = \{1, 2, \dots, d-1\}.$$

Dephasing: $V_j = V_j(H)$, for V_j some function in $\mathcal{D}(H)$.

- With $H = \sum_{k=1}^d e_k P_k$; $\mathcal{D}(H) = \{e_k\}_{k=1}^d$ simple, $P_k = |\psi_k\rangle\langle\psi_k|$; $\|\psi_k\|=1$.

$$\Rightarrow V_j = \sum_k v_j(k) P_k \quad \text{and} \quad V_j^* V_j = V_j V_j^* = \sum_k |v_j(k)|^2 P_k.$$

$$\begin{aligned} L(|\psi_r\rangle\langle\psi_s|) &= \left(-i(e_r - e_s) + \sum_j (v_j(r) \overline{v_j(s)} - \frac{1}{2} (|v_j(r)|^2 + |v_j(s)|^2)) \right) |\psi_r\rangle\langle\psi_s| \\ &\equiv \lambda_{rs} |\psi_r\rangle\langle\psi_s|. \end{aligned}$$

$$\lambda_{rs} = \overline{\lambda_{sr}} ; \quad \text{Re } \lambda_{rs} \leq 0 ; \quad \lambda_{rs} = 0 \Leftrightarrow r = s.$$

Lemma:

L diagonalisable, generically simple non zero eigenvalues,

and $\text{Ker } L = \text{span} \{P_h, h=1, \dots, d\} = \text{Ker } ([H, \cdot])$ of dim d .

$$P_0(f) = \sum_h P_h f P_h ; \quad Q_0 := I - P_0 \text{ s.t. } Q_0(f) = \sum_{j \neq h} P_j f P_h$$

Note: spectral proj onto $\text{Ker}(L - \lambda_{rs})$: $f \mapsto P_{rs}(f) = \langle \varphi_r | f \varphi_s \rangle \cdot |\varphi_r \rangle \langle \varphi_s|$

Theorem:

Assume $(L_t)_{t \in [0,1]}$ smooth time dep. generic dephasing Lindbladian.

Then $\varepsilon \dot{f}_\varepsilon = L_t(f_\varepsilon)$ admits a solution of the form

$$f_\varepsilon(t) = P_1(t) + \varepsilon \left(\sum_{j \neq 1} \frac{P_j(t) \dot{P}_1(t)}{\lambda_{j1}(t)} + \frac{\dot{P}_1(t) P_j(t)}{\lambda_{1j}(t)} - \sum_{j \neq 1} (P_1(t) - P_j(t)) \int_0^t \alpha_j(s) ds \right) + O(\varepsilon^2).$$

$$\alpha_j(t) = t_2 \left(P_1(t) \dot{P}_j(t)^2 P_1(s) \right) \cdot \frac{(-2 \operatorname{Re} \lambda_{1j}(t))}{|\lambda_{1j}(t)|^2} \geq 0. \quad (\text{Aren et al '12})$$

Proof: (method) Plug in the Ansatz , $(P_0(t) + Q_0(t) = 1)$

$$f_\varepsilon(t) = a_0(t) + b_0(t) + \varepsilon(a_1(t) + b_1(t)) + \varepsilon^2(a_2(t) + b_2(t)) + O(\varepsilon^3);$$

where $a_j(t) \in \text{Ran } P_0(t)$, $b_j(t) \in \text{Ran } Q_0(t)$.

$$\begin{aligned} \varepsilon(\dot{a}_0 + \dot{b}_0) + \varepsilon^2(\dot{a}_1 + \dot{b}_1) &= L(a_0) + L(b_0) + \varepsilon(L(a_1) + L(b_1)) + \varepsilon^2(L(a_2) + L(b_2)) \\ &\quad + O(\varepsilon^3) \\ \bullet \quad L P_0 = P_0 L &= 0 \quad \Rightarrow \quad L(a_j) = 0 \quad \forall j \end{aligned}$$

$$\varepsilon^0: L(b_0) = 0, \quad \varepsilon: \dot{a}_0 + \dot{b}_0 = L(b_1) \text{ and } \varepsilon^2: (\dot{a}_1 + \dot{b}_1) = L(b_2)$$

$$\bullet \quad L|_{Q_0 L(\mathbb{C}^d)} \text{ invertible, inverse } L^{-1}_{Q_0}: Q_0 L(\mathbb{C}^d) \rightarrow Q_0 L(\mathbb{C}^d)$$

$$Q_0(a) = 0 \Rightarrow \dot{Q}_0(a) + Q_0(\dot{a}) = -\dot{P}_0(a) + Q_0(\dot{a}) = 0$$

$$\text{i.e. } Q_0(\dot{a}) = \dot{P}_0(a) \quad \text{and} \quad P_0(\dot{b}) = -\dot{P}_0(b)$$

Application of Q_0 and P_0 to

- $\varepsilon^0: L(b_0) = 0$, $\varepsilon: \dot{a}_0 + \dot{b}_0 = L(b_1)$ and $\varepsilon^2: (\dot{a}_1 + \dot{b}_1) = L(b_2)$

yields $b_0 = 0$, $\dot{P}_0(a_h) + \dot{Q}_0(\dot{b}_h) = L(b_{h+1})$; $\dot{P}_0(\dot{a}_h) - \dot{P}_0(\dot{b}_h) = 0$. $h=0,1$.

Thus: (if b_h known)

- $\dot{a}_h = Q_0(\dot{a}_h) + P_0(\dot{a}_h) = \dot{P}_0(a_h) + \dot{P}_0(\dot{b}_h) = \underbrace{\dot{P}_0 P_0(a_h)}_{\equiv 0} - \dot{P}_0 \dot{P}_0(a_h) + \dot{P}_0(\dot{b}_h)$

Recall: $W(t,s)$ defined by $\partial_t W(t,s) = [\dot{\tilde{P}}_0(t), \dot{\tilde{P}}_0(t)] W(t,s)$; $W(s,s) = \mathbb{I}$.

$$\text{Da hamel: } a_h(t) = W(t,0) a_h(0) + \int_0^t W(t,s) \dot{\tilde{P}}_0(s)(b_h(s)) ds$$

$$\text{Juvenia: } b_{h+1}(t) = \tilde{L}_{Q_0(t)}^{-1} \left[\dot{\tilde{P}}_0(t)(a_h(t)) \right] + \tilde{L}_{Q_0(t)}^{-1} \left[\dot{\tilde{Q}}_0(t)(b_h(t)) \right]$$

Actually: the remainder term is $O(\varepsilon^{N+1})$, truncating after $N+1$ terms

Remains to compute:

• $b_0 \equiv 0$; $a_0 := P_1$ or. $\dot{P}_1 = \dot{P}_1 P_1 + P_1 \dot{P}_1 = \sum_j (\dot{P}_j P_1 P_j + P_j P_1 \dot{P}_j) \equiv \dot{P}_0(P_1)$.
i.e. $W(t, s) P_j(s) = P_j(t) \equiv P_0(t)(P_j(t))$.

$b_1 = \tilde{L}_{Q_0}^{-1} [\dot{P}_1 P_1 + P_1 \dot{P}_1] + 0 = \sum_{j \neq k} P_{jk} (\dot{P}_j P_k + P_j \dot{P}_k) / \lambda_{jk}$

$$P_{jk}(\cdot) = |\psi_j\rangle\langle\psi_k| \cdot \langle\psi_j|\cdot|\psi_k\rangle \quad \& \quad \dot{P}_1 = |\dot{\psi}_1\rangle\langle\psi_1| + |\psi_1\rangle\langle\dot{\psi}_1|$$

$$= \sum_{j \neq 1} \frac{P_j \dot{P}_1}{\lambda_{j1}} + \frac{\dot{P}_1 P_j}{\lambda_{1j}}$$

a_1 : $\dot{P}_0(b_1) = \sum_k \sum_{j \neq 1} \dot{P}_k \left\{ \frac{P_j \dot{P}_1}{\lambda_{j1}} + \frac{\dot{P}_1 P_j}{\lambda_{1j}} \right\} P_k + P_k \left\{ \text{same} \right\} \dot{P}_k$

$$\dot{P}_k P_j \dot{P}_1 P_k = \delta_{k1} \dot{P}_1 P_j \dot{P}_1 \quad ; \quad \dot{P}_k \dot{P}_1 P_j P_k = \delta_{jk} \dot{P}_j \dot{P}_1 P_j$$

$$\begin{aligned}\dot{\mathcal{P}}_o(h_1) &= \sum_{j \neq 1} \frac{\dot{P}_1 P_j \dot{P}_1}{\lambda_{j1}^2} + \frac{\dot{P}_j \dot{P}_1 P_j}{\lambda_{1j}} + \frac{P_j \dot{P}_1 \dot{P}_j}{\lambda_{j1}^2} + \frac{\dot{P}_1 P_j \dot{P}_1}{\lambda_{1j}^2} \\ &= \sum_{j \neq 1} \left(\underbrace{P_1 \dot{P}_1 P_j \dot{P}_1 P_1}_{P_1 |\langle \varphi_1 | \dot{P}_1 \varphi_j \rangle|^2} - \underbrace{P_j \dot{P}_j P_1 \dot{P}_j P_1}_{P_j |\langle \varphi_j | \dot{P}_1 \varphi_1 \rangle|^2} \right) \frac{2 \operatorname{Re} \lambda_{1j}}{|\lambda_{1j}|^2}\end{aligned}$$

But $\dot{P}_j = |\dot{\varphi}_j\rangle\langle\varphi_j| + |\varphi_j\rangle\langle\dot{\varphi}_j| \Rightarrow \langle\varphi_j | \dot{P}_j \varphi_1 \rangle = \langle\dot{\varphi}_j | \varphi_1 \rangle = -\langle\varphi_j | \dot{\varphi}_1 \rangle = -\langle\varphi_j | \dot{P}_1 \varphi_1 \rangle$

Thus $\dot{\mathcal{P}}_o(h_1) = \sum_{j \neq 1} (P_1 - P_j) \underbrace{|\langle \varphi_j | \dot{P}_1 \varphi_1 \rangle|}_{h_2(P_1 \dot{P}_j^2 P_1)}^2 \frac{2 \operatorname{Re} \lambda_{1j}}{|\lambda_{1j}|^2}, \quad \dot{P}_j = |\dot{\varphi}_j\rangle\langle\varphi_j| + |\varphi_j\rangle\langle\dot{\varphi}_j|$

Using $\int_0^t W(t,s) P_k(s) X_k(s) ds = P_k(t) \int_0^t X_k(s) ds, \text{ where } X_k(s) \in \mathbb{C}$,

and choosing $a_1(0) = 0$, we get

$$a_1(t) = \sum_{j \neq 1} (P_1(t) - P_j(t)) \int_0^t h_2(P_1(s) \dot{P}_j^2(s) P_1(s)) \frac{2 \operatorname{Re} \lambda_{1j}(s)}{|\lambda_{1j}(s)|^2} ds$$

□