

## Applications to the dynamics of Open Quantum Systems:

The dynamics of a small system of interest, weakly coupled to an environment can be described in the Markovian approximation by an effective lin. evol. equation:

$$\begin{cases} \dot{\rho}(t) = L(\rho(t)), & t \geq 0, \text{ where } \rho(t) \in \mathcal{T}(\mathcal{H}) \\ \rho(0) = \rho_0 & L : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}) \end{cases} \quad \begin{matrix} \text{is a density matrix} \\ \text{the Lindbladian} \end{matrix}$$

Actually, requirements from Physics are  $\forall t \geq 0$ ,

- $\rho(t) = e^{tL}(\rho_0) \geq 0$ ,  $\text{tr } \rho(t) \equiv 1$ ,  $e^{tL}$  is a CP (completely positive) map.

$$\left( T \in \mathcal{L}(\mathcal{L}(\mathcal{H})) \text{ CP : } \forall n \in \mathbb{N}^*, T \otimes \mathbb{I}_{id_m} \in \mathcal{L}(\mathcal{L}(\mathcal{H}) \otimes M_n(\mathbb{C})) \text{ is positive} \right)$$

Facts: •  $(e^{tL})_{t \geq 0}$  is a norm-continuous family of CPTP maps on  $\mathcal{L}(\mathcal{H})$

$$\Leftrightarrow L(f) = -i[H, f] + \sum_{j \in J} V_j f V_j^* - \frac{1}{2} \{ V_j^* V_j, f \}, \quad \forall f \in \mathcal{T}(\mathcal{H})$$

where  $H = H^* \in \mathcal{L}(\mathcal{H})$ ,  $V_j \in \mathcal{L}(\mathcal{H})$ ,  $\forall j \in J$ ,  $\sum_{j \in J} V_j^* V_j$  is a CV  
b.d.

Gini, Konakov, Sudarshan '76 (matrix case), Lindblad '76 (bounded case).

•  $H$  and  $\{V_j\}_{j \in J}$  are not unique.

•  $e^{tL}$  positive  $\Rightarrow \|e^{tL}\|_{op} = 1$ ,  $\forall t \geq 0$ , (contraction semigroup)  
as a map on  $\mathcal{T}(\mathcal{H})$ .

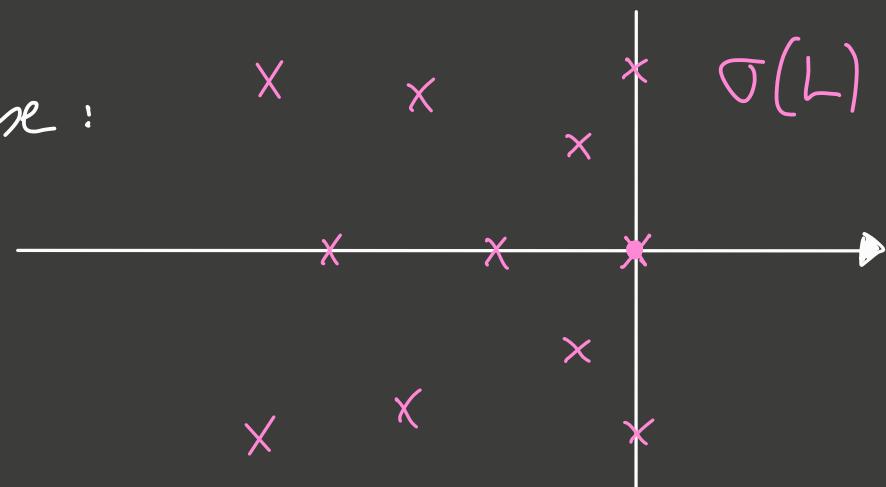
- Matrix case  $\mathcal{L}(\mathbb{C}^d)$  is a Hilbert space with  $(A, B) = \text{tr}(A^* B)$  as scal. prod.

$L \neq L^*$ , not normal.  $0 \in \sigma(L)$ ;  $\sigma(L) \subset \{Re z \leq 0\}$ .

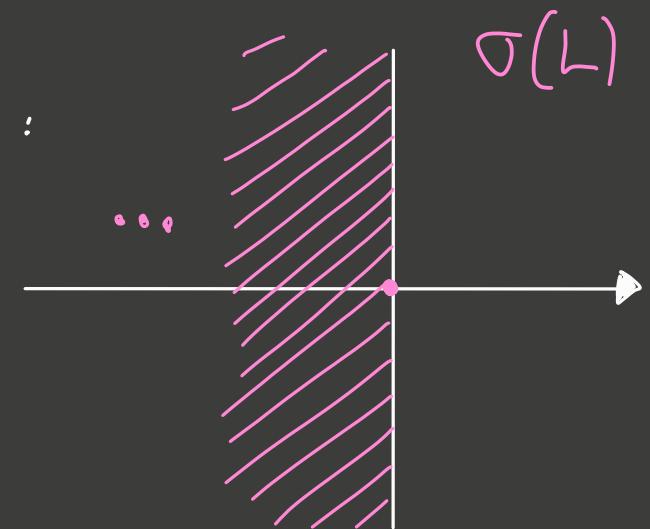
$\lambda \in \sigma(L)$  with  $Re \lambda = 0 \Rightarrow$  no associated eigen-multipotent

$$L(f^*) = (L(f))^* \Rightarrow \sigma(L) = \overline{\sigma(L)}$$

Matrix case:



In general:



- When  $H(t)$ ,  $V_j(t)$ ,  $j \in J$  are time dependent,  $L \mapsto L_t$  is the generator of a  $k_{\text{par}}$ -parameter evolution operator  $U(t, t_0)$  in  $\mathcal{T}(\mathcal{H})$ , if  $t \mapsto L_t$  schr-C°.

$$\forall t \geq t_0, \quad \partial_t U(t, t_0) f = L_t(U(t, t_0) f), \quad \forall f \in \mathcal{T}(\mathcal{H}); \quad U(t_0, t_0) = \mathbb{I},$$

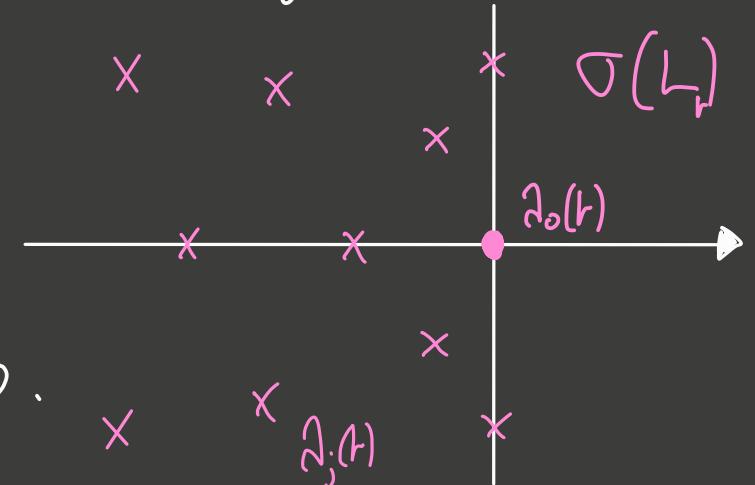
$$\text{o.b. } \|U(t, t_0) f\|_{\text{op}} \leq \|f\|_1, \quad \forall f \in \mathcal{T}(\mathcal{H}).$$

• A diabatic evolution:  $\varepsilon \dot{f} = L_t(f) ; t \in [0,1],$  assuming  $t \mapsto L_t$   $C^2.$

- Matrix case:  $\mathcal{H} = \mathbb{C}^d$

Assume  $\sigma(L_t) = \{\lambda_0(t), \lambda_1(t), \dots, \lambda_{d^2-1}(t)\}$  o.b.  $\text{mult.}(\lambda_j(t)) = m_j, \forall t \in [0,1].$

and  $L_t = \sum_{j=0}^N \lambda_j(t) P_j \in \mathcal{L}(M_d(\mathbb{C}))$  where  $N \leq d^2-1,$



i.e. no Jordan blocs.  $\lambda_0(t) \equiv 0, \operatorname{Re} \lambda_j(t) \leq 0.$

Then,  $f_\varepsilon(t)$  sol. to  $\varepsilon \dot{f}_\varepsilon = L_t(f_\varepsilon), f_\varepsilon(0) = f = P_0(f) \in M_d(\mathbb{C})$  satisfies

$f_\varepsilon(t) = W(t) P_0(f) + O(\varepsilon),$  uniformly in  $t \in [0,1],$  where

$W(t)$  defined by  $W'(t) = \left( \sum_{j=0}^N P_j'(t) P_j(t) \right) W(t); W(0) = \mathbb{1}$  o.b.

$W(t) P_j(0) = P_j(t) W(t), \forall t \in [0,1], \forall j \in \{0, \dots, N\}.$

Proof:

i) the intertwining property of  $\omega$  is similar to the case considered so far.

$$\text{ii) } \sigma(iL_r) = \{i\lambda_j(r)\}_{j=0}^N \quad \text{with } \operatorname{Im}(i\lambda_j(r)) \leq 0.$$

$$\text{iii) } \phi_\varepsilon(t) \text{ s.t. } i\varepsilon \phi'_\varepsilon(t) = \left( \tilde{W}(t) iL_r W(t) \right) \phi_\varepsilon(t); \quad \phi_\varepsilon(0) = 1$$

$$\phi_\varepsilon(t) = \sum_{j=0}^N P_j(0) e^{\frac{1}{\varepsilon} \int_0^t \lambda_j(s) ds} = P_0(0) + \sum_{j>0} P_j(0) e^{\frac{1}{\varepsilon} \int_0^t \lambda_j(s) ds}.$$

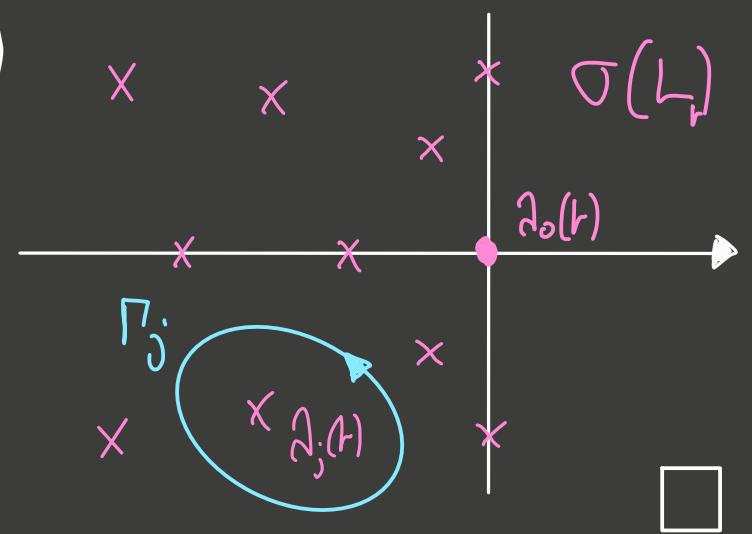
s.t.  $\sup_{t \in [0,1]} \sup_{\varepsilon > 0} \|\phi_\varepsilon^{\pm 1}(t)\| < C_1 \Rightarrow \sup_{t \in [0,1]} \sup_{\varepsilon > 0} \|V_\varepsilon^{\pm 1}(t)\| \leq C_2$

(b) Integration by parts on  $\sum_j P'_j S_j$  in place of  $[P', P]$  using

$$\mathcal{R}_j(B) := -\frac{1}{2i\pi} \int_{\Gamma_j} (H-z)^{-1} B (H-z)^{-1} dz \quad (\text{with } H=iL) \quad \times \quad \times \quad \times \quad \sigma(L)$$

$$P'_j S_j = [P'_j S_j, S_j] = [H, \mathcal{R}_j(P'_j S_j)]$$

$$\text{v) } U_\varepsilon(t) = V_\varepsilon(t) + O(\varepsilon).$$



□

Remarks:

- Generically,  $\text{mult}(\mathcal{J}_j(t)) = 1$ ; In any case, no nilpotent associated to  $\mathcal{J}_0(t) \equiv 0$ .
- In case nilpotents are present, see J. '08 (exponentially accurate)
- For adiabatic evolution for generators of contractions in Banach spaces,  
see Aron Fraas Graf Grech '12, J. '08 (with and without gap).
- Relevant references (past and present):  
Nenciu-Raschke '92, Abou-Salem, Fröhlich '05-'06, Sarandy, Lidar '05  
Aron Fraas Graf Grech '11, Aron Fraas Graf '12, Fraas, Hänggli '17,
- Nice application of the matrix case to dephasing Lindbladians, see below.

- Dephasing Lindbladians:

Auron et al '12

$\mathcal{H} = \mathbb{C}^d$ ,  $\mathcal{L}(\mathbb{C}^d)$  Hilbert with  $(A, B) = \text{Tr}(A^* B)$ .

$$L(f) = -i[H, f] + \sum_{j \in J} V_j f V_j^* - \frac{1}{2} \left\{ V_j^* V_j, f \right\} ; \quad J = \{1, 2, \dots, d-1\}.$$

Dephasing:  $V_j = V_j(H)$ , for  $V_j$  some function in  $\mathcal{D}(H)$ .

- With  $H = \sum_{k=1}^d e_k P_k$ ;  $\mathcal{D}(H) = \{e_k\}_{k=1}^d$  simple,  $P_k = |\varphi_k\rangle\langle\varphi_k|$ ;  $\|\varphi_k\|=1$ .

$$\Rightarrow V_j = \sum_k v_j(k) P_k \quad \text{and} \quad V_j^* V_j = V_j V_j^* = \sum_k |v_j(k)|^2 P_k.$$

$$\begin{aligned} L(|\varphi_r\rangle\langle\varphi_s|) &= \left( -i(e_r - e_s) + \sum_j \left( v_j(r) \overline{v_j(s)} - \frac{1}{2} (|v_j(r)|^2 + |v_j(s)|^2) \right) \right) |\varphi_r\rangle\langle\varphi_s| \\ &\equiv \lambda_{rs} |\varphi_r\rangle\langle\varphi_s|. \end{aligned}$$

$$\lambda_{rs} = \overline{\lambda_{sr}} ; \quad \operatorname{Re} \lambda_{rs} \leq 0 ; \quad \lambda_{rs} = 0 \Leftrightarrow r = s.$$

Lemma:

$L$  diagonalisable, generically simple non zero eigenvalues,

and  $\text{Ker } L = \text{span} \{P_h, h=1, \dots, d\} = \text{Ker } ([H, \cdot])$  of dim  $d$ .

$$P_0(f) = \sum_{h=1}^d P_h f P_h ; \quad Q_0 := I - P_0 \text{ s.t. } Q_0(f) = \sum_{j \neq h} P_j f P_h$$

Note: spectral proj onto  $\text{Ker}(L - \mathcal{J}_{rs})$ :  $f \mapsto P_{rs}(f) = \langle \varphi_r | f \varphi_s \rangle \cdot |\varphi_r \rangle \langle \varphi_s|$

Theorem:

Assume  $(L_t)_{t \in [0,1]}$  smooth time dep. generic dephasing Lindbladian.

Then  $\varepsilon \dot{f}_\varepsilon = L_t(f_\varepsilon)$  admits a solution of the form

$$f_\varepsilon(t) = P_1(t) + \varepsilon \left( \sum_{j \neq 1} \frac{P_j(t) \dot{P}_1(t)}{\mathcal{J}_{j1}(t)} + \frac{\dot{P}_1(t) P_j(t)}{\mathcal{J}_{1j}(t)} - \sum_{j \neq 1} (P_1(t) - P_j(t)) \int_0^t \mathcal{L}_j(s) ds \right) + O(\varepsilon^2).$$

$$\mathcal{L}_j(t) = t_2 \left( P_1(t) \dot{P}_j(t)^2 P_1(s) \right) \cdot \frac{(-2 \operatorname{Re} \mathcal{J}_{1j}(t))}{|\mathcal{J}_{1j}(t)|^2} \geq 0.$$

(Aren et al '12)

Proof: (method) Plug in the Ansatz ,  $(P_0(t) + Q_0(t) = 1)$

$$f_\varepsilon(t) = a_0(t) + b_0(t) + \varepsilon(a_1(t) + b_1(t)) + \varepsilon^2(a_2(t) + b_2(t)) + O(\varepsilon^3);$$

where  $a_j(t) \in \text{Ran } P_0(t)$ ,  $b_j(t) \in \text{Ran } Q_0(t)$ .

$$\begin{aligned} \varepsilon(\dot{a}_0 + \dot{b}_0) + \varepsilon^2(\dot{a}_1 + \dot{b}_1) &= L(a_0) + L(b_0) + \varepsilon(L(a_1) + L(b_1)) + \varepsilon^2(L(a_2) + L(b_2)) \\ &\quad + O(\varepsilon^3) \end{aligned}$$

•  $L P_0 = P_0 L = 0 \Rightarrow L(a_j) = 0 \quad \forall j$

$$\varepsilon^0: L(b_0) = 0, \quad \varepsilon: \dot{a}_0 + \dot{b}_0 = L(b_1) \text{ and } \varepsilon^2: (\dot{a}_1 + \dot{b}_1) = L(b_2)$$

•  $L|_{Q_0 \mathcal{L}(\mathbb{C}^d)}$  invertible, inverse  $\bar{L}_{Q_0}^{-1}: Q_0 \mathcal{L}(\mathbb{C}^d) \rightarrow Q_0 \mathcal{L}(\mathbb{C}^d)$

$$Q_0(a) = 0 \Rightarrow \dot{Q}_0(a) + Q_0(\dot{a}) = -\dot{P}_0(a) + Q_0(\dot{a}) = 0$$

i.e.  $Q_0(\dot{a}) = \dot{P}_0(a)$  and  $P_0(\dot{b}) = -\dot{P}_0(b)$

Application of  $Q_0$  and  $P_0$  to

- $\varepsilon^0$ :  $L(b_0) = 0$ ,  $\varepsilon$ :  $\dot{a}_0 + \dot{b}_0 = L(b_1)$  and  $\varepsilon^2$ :  $(\dot{a}_1 + \dot{b}_1) = L(b_2)$

yields  $b_0 = 0$ ,  $\dot{P}_0(a_h) + Q_0(\dot{b}_h) = L(b_{h+1})$ ;  $\dot{P}_0(\dot{a}_h) - \dot{P}_0(\dot{b}_h) = 0$ .  $h=0,1$ .

Thus: (if  $b_h$  known)

- $\dot{a}_h = Q_0(\dot{a}_h) + P_0(\dot{a}_h) = \dot{P}_0(a_h) + \dot{P}_0(b_h) = \underbrace{\dot{P}_0 P_0(a_h)}_{\equiv 0} - \dot{P}_0 \dot{P}_0(a_h) + \dot{P}_0(b_h)$

Recall:  $W(t,s)$  defined by  $\partial_t W(t,s) = [\dot{P}_0(t), \dot{P}_0(s)] W(t,s)$ ;  $W(s,s) = \mathbb{I}$ .

Duhamel:  $a_h(t) = W(t,0) a_h(0) + \int_0^t W(t,s) \dot{P}_0(s)(b_h(s)) ds$

Inversion:  $b_{h+1}(t) = \mathcal{L}_{Q_0(t)}^{-1} [\dot{P}_0(t)(a_h(t))] + \mathcal{L}_{Q_0(t)}^{-1} [Q_0(t)(\dot{b}_h(t))]$  General!

Actually: the remainder term is  $O(\varepsilon^{N+1})$ , truncating after  $N+1$  terms

Remains to compute:

- $b_0 \equiv 0$  ;  $a_0 := P_1$  or.  $\dot{P}_1 = \dot{P}_1 P_1 + P_1 \dot{P}_1 = \sum_j \left( \dot{P}_j P_1 P_j + P_j P_1 \dot{P}_j \right) \equiv \dot{P}(P_1)$ .  
i.e.  $W(t, s) P_j(s) = P_j(t) \equiv P_0(t)(P_j(t))$ .

$$\underline{b_1} = \bar{L}_{Q_0}^{-1} [\dot{P}_1 P_1 + P_1 \dot{P}_1] + 0 = \sum_{j \neq h} P_{jh} (\dot{P}_1 P_1 + P_1 \dot{P}_1) / \lambda_{jh}$$

$$P_{jh}(\cdot) = |\psi_j\rangle\langle\psi_h| \cdot \langle\psi_j|\cdot\psi_h\rangle \quad \& \quad \dot{P}_1 = |\dot{\psi}_1\rangle\langle\psi_1| + |\psi_1\rangle\langle\dot{\psi}_1|$$

$$= \sum_{j \neq 1} \frac{P_j \dot{P}_1}{\lambda_{j1}} + \frac{\dot{P}_1 P_j}{\lambda_{1j}}$$

$$\underline{a_1}: \quad \dot{P}_0(b_1) = \sum_h \sum_{j \neq 1} \dot{P}_h \left\{ \frac{P_j \dot{P}_1}{\lambda_{j1}} + \frac{\dot{P}_1 P_j}{\lambda_{1j}} \right\} P_h + P_h \left\{ \text{same} \right\} \dot{P}_h$$

$$\dot{P}_h P_j \dot{P}_1 P_h = \delta_{h1} \quad \dot{P}_1 P_j \dot{P}_1 = \delta_{jh} \quad ; \quad \dot{P}_h \dot{P}_1 P_j P_h = \delta_{jh} \quad \dot{P}_j \dot{P}_1 P_j$$

$$\begin{aligned}\dot{\mathcal{P}}_0(h_1) &= \sum_{j \neq 1} \frac{\dot{P}_1 P_j \dot{P}_1}{\lambda_{j1}^2} + \frac{\dot{P}_j \dot{P}_1 P_j}{\lambda_{1j}} + \frac{P_j \dot{P}_1 \dot{P}_j}{\lambda_{j1}} + \frac{\dot{P}_1 P_j \dot{P}_1}{\lambda_{1j}} \\ &= \sum_{j \neq 1} \left( \underbrace{P_1 \dot{P}_1 P_j \dot{P}_1 P_1}_{P_1 |\langle \varphi_1 | \dot{P}_1 \varphi_j \rangle|^2} - \underbrace{P_j \dot{P}_j P_1 \dot{P}_j P_j}_{P_j |\langle \varphi_j | \dot{P}_j \varphi_1 \rangle|^2} \right) \frac{2 \operatorname{Re} \lambda_{1j}}{|\lambda_{1j}|^2}\end{aligned}$$

But  $\dot{P}_j = |\dot{\varphi}_j\rangle\langle\varphi_j| + |\varphi_j\rangle\langle\dot{\varphi}_j| \Rightarrow \langle\varphi_j | \dot{P}_j \varphi_1 \rangle = \langle\dot{\varphi}_j | \varphi_1 \rangle = -\langle\varphi_j | \dot{\varphi}_1 \rangle = -\langle\varphi_j | \dot{P}_j \varphi_1 \rangle$

Thus  $\dot{\mathcal{P}}_0(h_1) = \sum_{j \neq 1} (P_1 - P_j) \underbrace{|\langle \varphi_j | \dot{P}_j \varphi_1 \rangle|^2}_{h_2(P_1 \dot{P}_j^2 P_1)} \frac{2 \operatorname{Re} \lambda_{1j}}{|\lambda_{1j}|^2}, \quad \dot{P}_j = |\dot{\varphi}_j\rangle\langle\varphi_j| + |\varphi_j\rangle\langle\dot{\varphi}_j|$

Using  $\int_0^t W(t,s) P_h(s) \chi_h(s) ds = P_h(t) \int_0^t \chi_h(s) ds, \text{ where } \chi_h(s) \in \mathbb{C},$

and choosing  $\chi_h(0) = 0,$  we get

$$a_1(t) = \sum_{j \neq 1} (P_1(t) - P_j(t)) \int_0^t h_2(P_1(s) \dot{P}_j^2(s) P_1(s)) \frac{2 \operatorname{Re} \lambda_{1j}(s)}{|\lambda_{1j}(s)|^2} ds$$

□