

Grenoble
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Adiabatic quantum dynamics and applications

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(Tentative) Program :

1) Adiabatic Theorems in Quantum Mechanics

- "à la Kato"
- superadiabatic
- exponential
- gapless

2) Applications to open quantum systems

- Lindbladians
- dephasing Lindbladians
- Two-level system in a Box field

Two-level system:

$S_\alpha \ni z \mapsto H(z) \in M_2(\mathbb{C})$, analytic.

$\forall z \in S_\alpha, H(\bar{z}) = H(z)^*$.

$\exists H(\pm\infty) = H(\pm\infty)^*$ s.t. $\sup_{|t| \leq \alpha} \|H(t+is) - H(\pm\infty)\| \leq b(t) \xrightarrow[t \rightarrow \pm\infty]{} 0$, $\int_{\mathbb{R}} b(t) dt < \infty$.

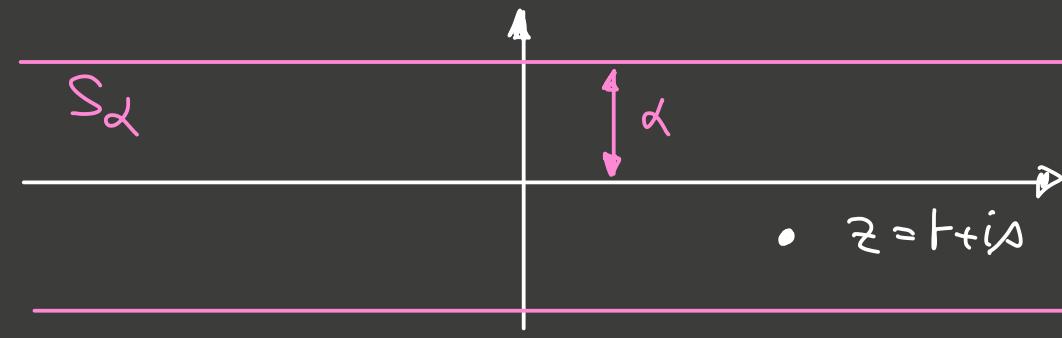
$\forall t \in \mathbb{R}, H(t) = \sum_j e_j(t) P_j(t)$,

s.t. $\{e_j(t)\}_j = \sigma(H(t)) \subset \mathbb{R}$,

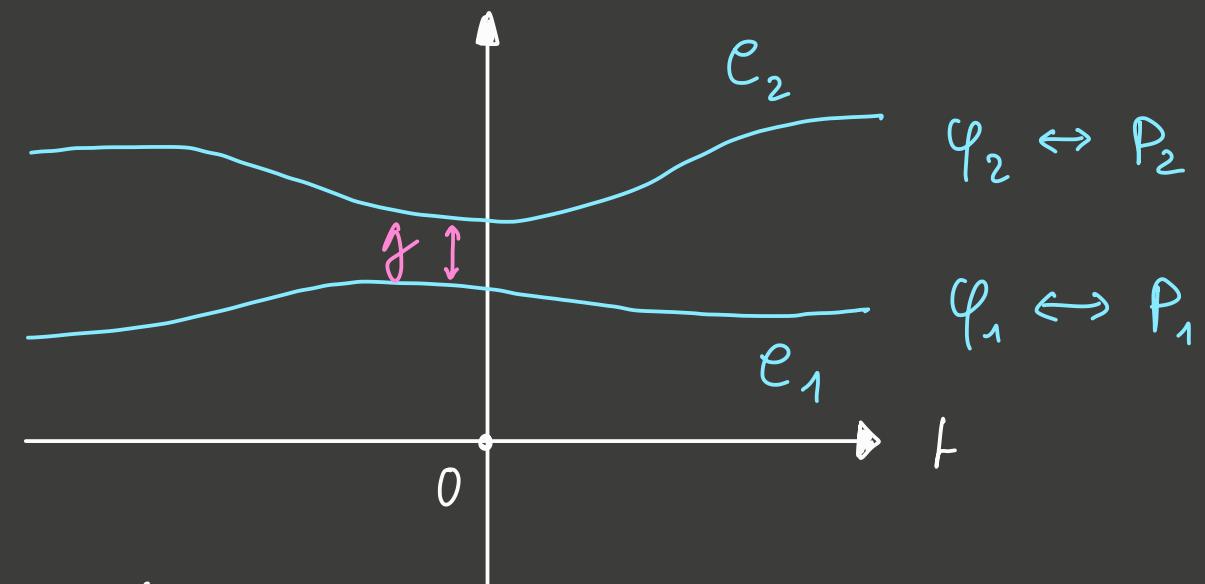
$$\inf_{t \in \mathbb{R}} e_2(t) - e_1(t) = g > 0.$$

$P_j(t) = |\varphi_j(t)\rangle\langle\varphi_j(t)| ; \|\varphi_j(t)\| = 1 ; \varphi_j \in C^\infty(\mathbb{R})$.

$$s.t. \quad \langle \varphi_j(t) | \varphi_j'(t) \rangle = 0$$



C.



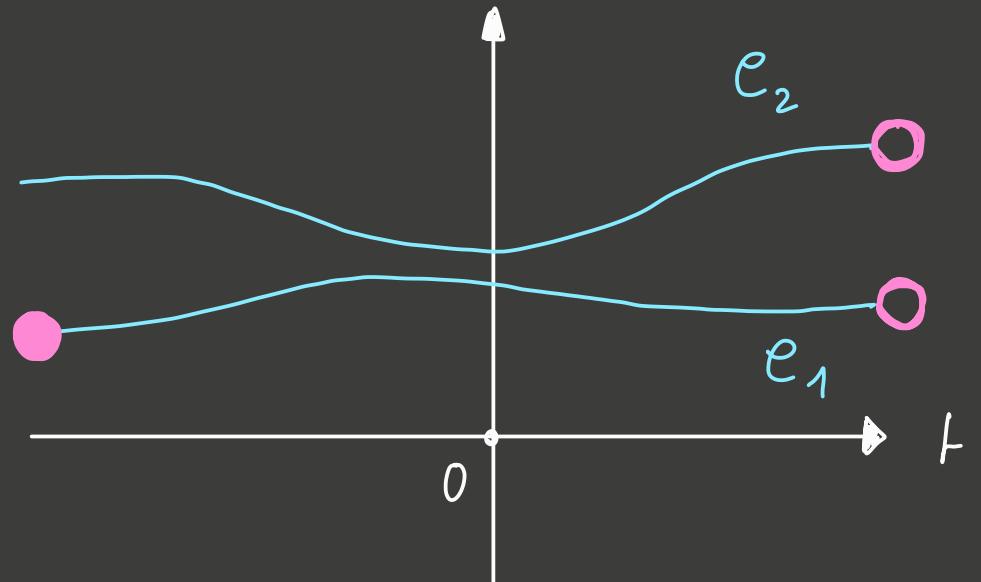
$$\Leftrightarrow \varphi_j(t) = W(t) \varphi_j(0), \text{ where } W(t) = [P_1'(t), P_1(t)] W(t) ; \quad W(0) = \mathbb{1}.$$

$$i\varepsilon \partial_t \Psi_\varepsilon = H(t) \Psi_\varepsilon, \quad t \in \mathbb{R}, \quad \|\Psi_\varepsilon(t)\| = 1.$$

$$\lim_{t \rightarrow -\infty} P_2(t) \Psi_\varepsilon(t) = 0$$

Q: $\lim_{t \rightarrow +\infty} P_2(t) \Psi_\varepsilon(t) = ?$

$$\Psi_\varepsilon(t) = \sum_{j=1}^2 C_j^\varepsilon(t) e^{-\frac{i}{\varepsilon} \int_0^t e_j(s) ds} \varphi_j(t) \quad \text{o.f.} \quad \lim_{t \rightarrow -\infty} C_1^\varepsilon(t) = 1; \quad |C_1^\varepsilon(t)|^2 + |C_2^\varepsilon(t)|^2 = 1.$$



Q $\Leftrightarrow \lim_{t \rightarrow +\infty} C_2^\varepsilon(t) = ?$ // Transport

$$\begin{cases} C_j^\varepsilon'(t) = a_{jk}(t) e^{+\frac{i}{\varepsilon} \int_0^t (e_j(s) - e_k(s)) ds} C_k^\varepsilon(t); & C_j^\varepsilon(-\infty) = \delta_{j,1}, \quad j \neq k. \\ a_{jk}(t) = -\langle \varphi_j(t) | \varphi_k'(t) \rangle = -\langle \varphi_j(0) | \tilde{W}(t) [P_j'(t), P_k(t)] W(t) \varphi_k(0) \rangle = O(b(t)) \end{cases}$$

Note: $\lim_{t \rightarrow \pm \infty} C_j^\varepsilon(t) := C_j^\varepsilon(\pm \infty)$ exists since $\int_{\mathbb{R}} b(t) dt < \infty$.

Analytic continuation:

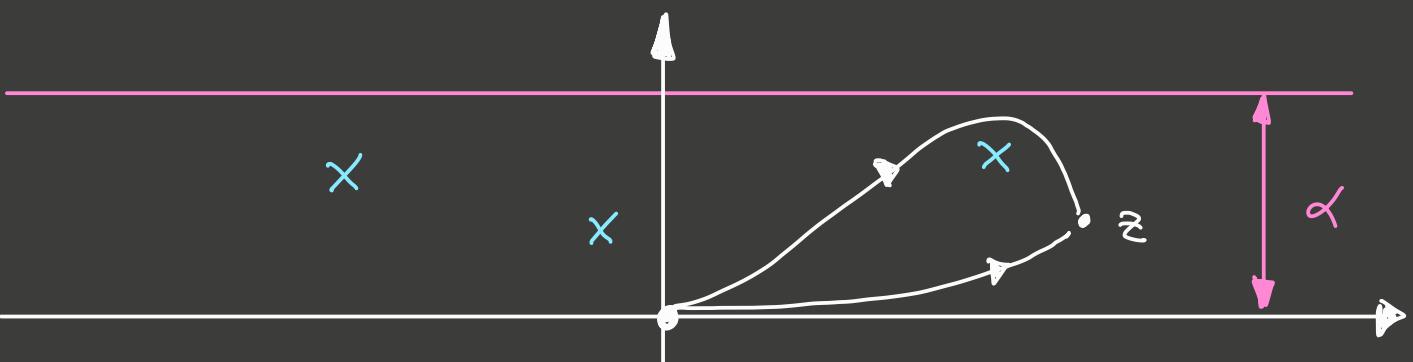
H is analytic in $S_\alpha \Rightarrow \Psi_\varepsilon$ has an analytic extension in S_α (sol. of ODE).

For $z \in S_\alpha$, charact. eq. $\lambda^2 - 2\operatorname{tr} H(z) + \det H(z) = 0$ with analytic coeff.

$\Rightarrow c_j$ have analytic extensions in $S_\alpha \setminus X$, where $X := \{z \in S_\alpha \text{ s.t. } c_1(z) = c_2(z)\}$.

c_j have at worse algebraic singularities at X (the set of turning points)

- X is finite (scatt. cond.),
- $X \cap \mathbb{R} = \emptyset$ (gap).



$\Rightarrow \rho_j$ have analytic extensions in $S_\alpha \setminus X$, and W , hence φ_j also.

Note: These extensions depend on the way to reach $z \in S_\alpha$ from 0. (Monodromy)

- c_j^ε admit analytic extensions from \mathbb{R} to $S_\alpha \setminus X$ that we shall exhibit.

Specifically for 2-level case:

$$z_0 \in X \Leftrightarrow \left(\text{tr } H(z_0) \right)^2 - 4 \det H(z_0) = 0$$

Generically: $e_2(z) - e_1(z) = \left\{ \left(\text{tr } H(z) \right)^2 - 4 \det H(z) \right\}^{1/2} = C_{z_0} (z - z_0)^{\frac{1}{k_2}} \left(1 + O(z - z_0) \right).$

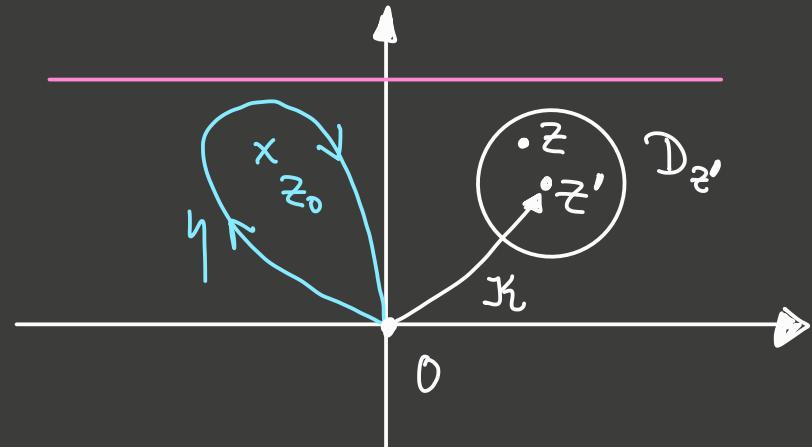
Also: $P_1(z), P_2(z)$ analytic in $S_2 \setminus X$ with (generic) singularities $\sim (z - z_0)^{-\frac{1}{k_2}}$

Notations: Let $\gamma \subset S_2$ be a simple loop with base point at 0, negatively oriented. Let $z' \in S_2 \setminus X$, $D_{z'} \subset S_2 \setminus X$ be a neighborhood of z' .

H f analytic in $S_2 \setminus X$, originally defined in D_0 ,

let $z \mapsto f(z)^\kappa$ its extension along κ in $D_{z'}$.

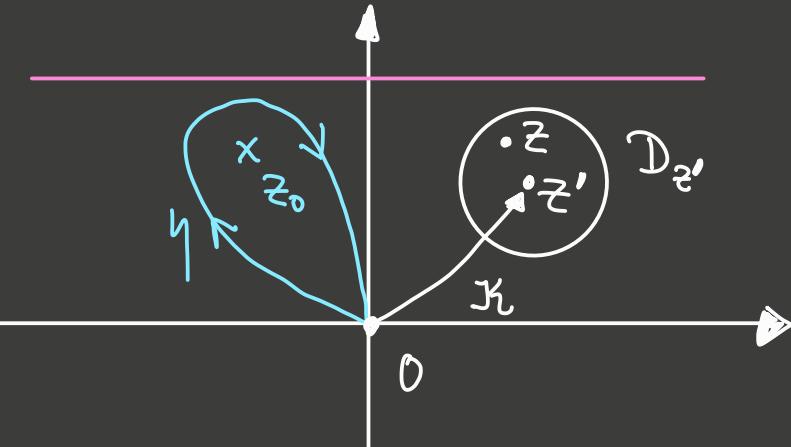
Set $z \mapsto f(z|\gamma)^\kappa$ the extension in $D_{z'}$ obtained along $\gamma \cdot \kappa$ (first γ , then κ)



Consequence: Let $z_0 \in X$ be a simple zero if $z \mapsto (\det H_z)^2 - 4 \det(H_z)$

$$j \neq k \in \{1, 2\}$$

- $e_j^{\kappa}(z|\gamma) = c_k^{\kappa}(z), \quad (\text{exchange of eigenvalues})$



- $\varphi_j^{\kappa}(z|\gamma) = e^{-i\phi_j(\gamma)} \varphi_k^{\kappa}(z), \quad \phi_j(\gamma) \in \mathbb{C}. \quad (\text{exchange of eigenvectors})$

- $\Psi_{\varepsilon}^{\kappa}(z) = \Psi_{\varepsilon}(z)$ is indep. of κ , since $H(z)$ analytic in S_d .

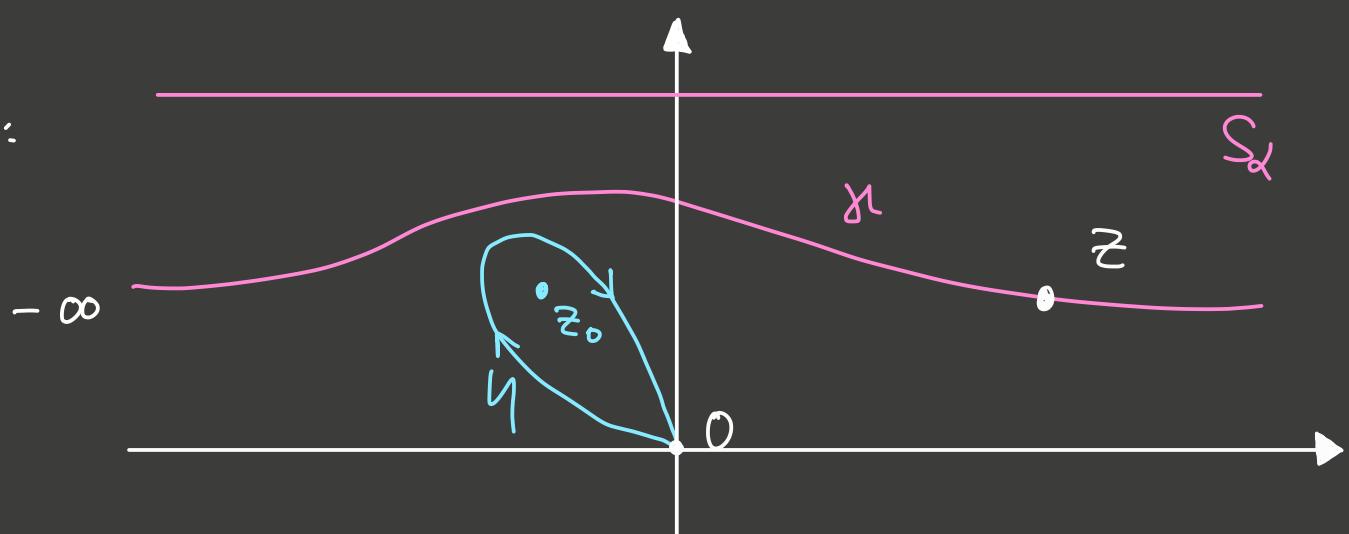
$$\Psi_{\varepsilon}(z) = \sum_{j=1}^2 c_j^{\kappa}(z) e^{-\frac{i}{\varepsilon} \left(\int_0^z e_j(s) ds \right)^{\kappa}} \varphi_j^{\kappa}(z) \equiv \sum_{j=1}^2 c_j^{\kappa}(z|\gamma) e^{-\frac{i}{\varepsilon} \left(\int_0^z e_j(s) ds \right)^{\kappa}} \varphi_j^{\kappa}(z|\gamma)$$

Proposition: $c_2^{\kappa}(z) = e^{-i\phi_1(\gamma)} e^{-\frac{i}{\varepsilon} \int_{\gamma} e_1(s) ds} c_1^{\kappa}(z|\gamma) \quad \forall z \in D_{z'}$

Control of $C_1(+\infty|\gamma)$ to get $C_2(+\infty)$:

$$C_1(-\infty) = 1$$

$$C_2(-\infty) = 0$$



$$\begin{cases} C_1(z) = 1 + \int_{-\infty}^z a_{12}(u) e^{+\frac{i}{\varepsilon} \int_0^u (e_1 - e_2)(s) ds} C_2(u) du \\ C_2(z) = \int_{-\infty}^z a_{21}(u) e^{+\frac{i}{\varepsilon} \int_0^u (e_2 - e_1)(s) ds} C_1(u) du \end{cases}$$

Rem: $\lim_{|t| \rightarrow \infty} c_j(t+is) = c_j(\pm\infty)$.

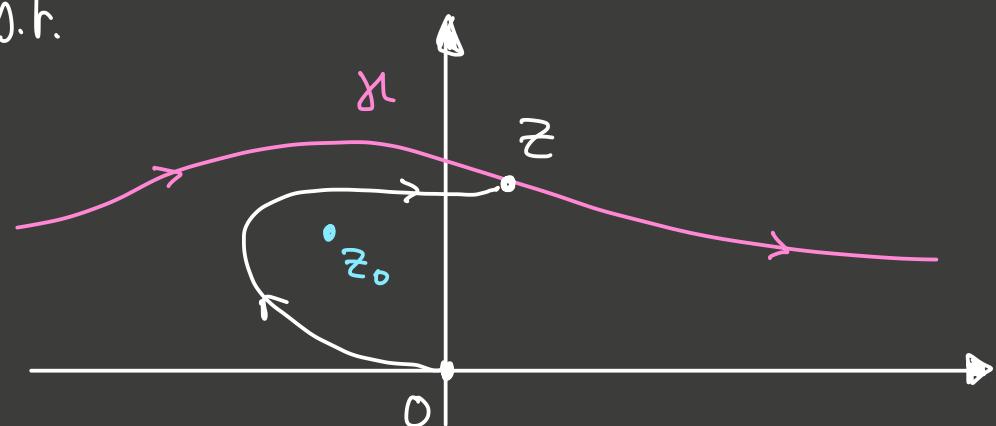
unif in $A \in [-d, d]$

Writing c_j the solution on \mathbb{R} , the monodromy relation yields

$$C_2(+\infty) = e^{-i\phi_1(\gamma)} e^{-\frac{i}{\varepsilon} \int_\gamma e_1(s) ds} C_1(+\infty)$$

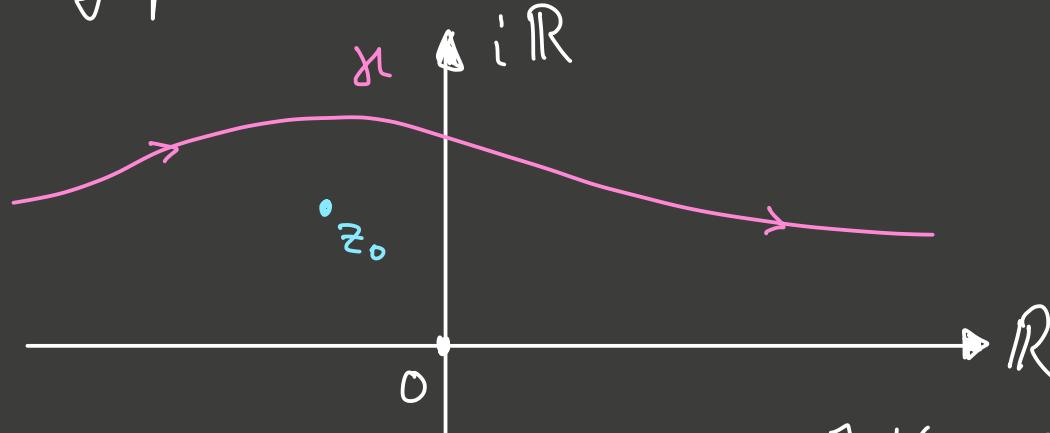
Divergent path: γ_1 , from $-\infty$ to $+\infty$, s.t.

$\operatorname{Im} \int_0^z (e_1 - e_2)(\tau) d\tau$ along γ_1



Theorem: Under the regularity and gap assumptions as $z \mapsto H(z)$, if

- $\exists \gamma$ a dissipative path
- $\exists! z_0 \in X$, generic o.t. :



J-Kunz-Pfister '91

Then: $i\varepsilon \partial_t \Psi_\varepsilon = H(t) \Psi_\varepsilon$, $t \in \mathbb{R}$, $\|\Psi_\varepsilon(t)\| \leq 1$, with $\lim_{t \rightarrow -\infty} P_2(t) \Psi_\varepsilon(t) = 0$

$$\Rightarrow \lim_{t \rightarrow +\infty} e^{+\frac{i}{\varepsilon} \int_0^t e_2(s) ds} \langle \varphi_2(t) | \Psi_\varepsilon(t) \rangle = e^{-i\vartheta_1(\eta)} e^{-\frac{i}{\varepsilon} \int_\eta^\infty e_1(s) ds} (1 + O(\varepsilon))$$

Nocality $\rightarrow |\cdot| \neq 1$ $|\cdot| = e^{-\eta/\varepsilon}$; $\eta > 0$.

Rem: • This "Dykhne formula" yields Landau-Zener for small gaps.

• Landau-Zener : $\begin{pmatrix} t & \delta \\ \delta & -t \end{pmatrix} \rightsquigarrow \Psi_\varepsilon(t)$ via special fcts.

$$\Rightarrow \lim_{t \rightarrow +\infty} \|P_2(t) \Psi_\varepsilon(t)\| = e^{-\frac{\pi i}{4} \frac{\delta^2}{\varepsilon}}, \quad \forall \delta, \varepsilon > 0.$$

• Zener '32, Majorana '32, Landau '32, Dykhne '62, Hwang-Pechukas '77, Beny '90

Proof: It is enough to show that $C_1^\infty(+\infty) = 1 + O(\varepsilon)$.

$$C_1^\infty(z) = 1 + \int_{-\infty}^z a_{12}(u) e^{+\frac{i}{\varepsilon} \int_0^u (e_1 - e_2)(s) ds} C_2^\infty(u) du$$

$$C_2^\infty(z) = \int_{-\infty}^z a_{21}(u) e^{+\frac{i}{\varepsilon} \int_0^u (e_2 - e_1)(s) ds} C_1^\infty(u) du$$

$\underbrace{\frac{1}{i(e_2 - e_1)(u)} \frac{d}{du} \left(e^{\frac{i}{\varepsilon} \int_0^u (e_2 - e_1)(s) ds} \right)}$

$$= \frac{\varepsilon}{i(e_2 - e_1)(z)} e^{+\frac{i}{\varepsilon} \int_0^z (e_2 - e_1)(s) ds} C_1^\infty(z) - 0 - \frac{\varepsilon}{i} \int_{-\infty}^z \frac{d}{du} \left(\frac{a_{21}(u)}{(e_2 - e_1)(u)} \right) e^{+\frac{i}{\varepsilon} \int_0^u (e_2 - e_1)(s) ds} C_1^\infty(u) du$$

$$- \frac{\varepsilon}{i} \int_{-\infty}^z \frac{a_{21}(u)}{(e_2 - e_1)(u)} a_{12}(u) C_2^\infty(u) du \quad \gtrsim \quad \text{from } \frac{d}{du} C_1^\infty(u)$$

New variables: $X_1(z) := C_1^\infty(z)$; $X_2(z) := e^{+\frac{i}{\varepsilon} \int_0^z (e_1 - e_2)(s) ds} C_2^\infty(z)$ so that

$$X_1(z) = 1 + \int_{-\infty}^z a_{12}(u) X_2(u) du$$

$$X_2(z) = \varepsilon \left\{ A(z) X_1(z) + \int_{-\infty}^z B(u) e^{\frac{i}{\varepsilon} \int_u^z (e_1 - e_2)(s) ds} X_1(u) du + \int_{-\infty}^z C(u) e^{\frac{i}{\varepsilon} \int_u^z (e_1 - e_2)(s) ds} X_2(u) du \right\}$$

where

Satt. hypoth: $\exists c < \infty$ s.t. $\max \left\{ |A(z)|, \int_{-\infty}^z |\beta(u)| du, \int_{-\infty}^z |\gamma(u)| du, \int_{-\infty}^z |a_{12}(u)| du \right\} \leq c$

Y dissipative: $\left| e^{\frac{i}{\varepsilon} \int_u^z (\epsilon_1 - \epsilon_2)(s) ds} \right| = e^{-\frac{1}{\varepsilon} \operatorname{Im} \int_u^z (\epsilon_1 - \epsilon_2)(s) ds} \leq 1 \quad \forall \varepsilon > 0.$

$$\Rightarrow |X_1(z)| \leq 1 + c \sup_{u \in Y} |X_2(u)|$$

$$|X_2(z)| \leq \varepsilon c \left(\sup_{u \in Y} |X_1(u)| + \sup_{u \in Y} |X_2(u)| \right)$$

and

$$\sup_{u \in Y} |X_1(u)| \leq 1 + c \sup_{u \in Y} |X_2(u)|$$

$$\sup_{u \in Y} |X_2(u)| \leq \varepsilon c \left(1 + (c+1) \sup_{u \in Y} |X_2(u)| \right).$$

Thus $\sup_{u \in Y} |X_2(u)| \left(1 - \varepsilon c(c+1) \right) \leq \varepsilon c \quad \xrightarrow{\varepsilon \text{ small}} \sup_{u \in Y} |X_2(u)| \leq \varepsilon \tilde{c}$. constant.

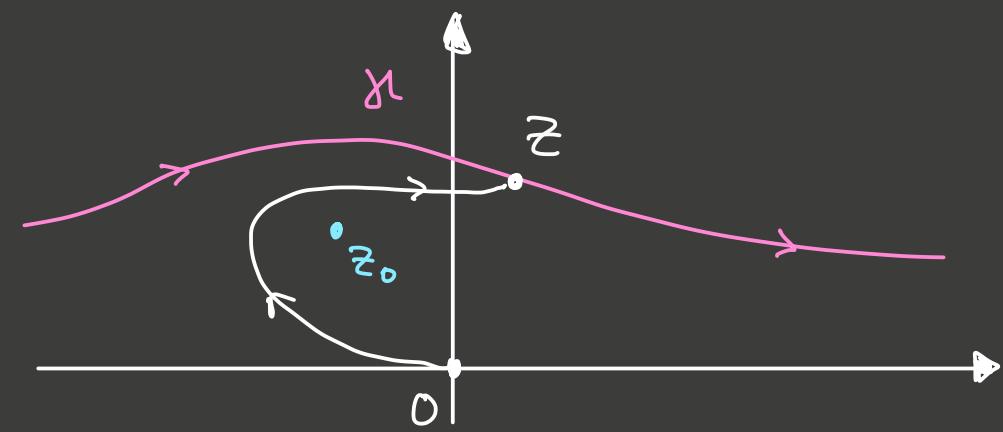
Finally: $|X_1(z) - 1| \leq c \sup_{u \in Y} |X_2(u)| \leq \varepsilon c \tilde{c}$, unif in $z \in Y$.



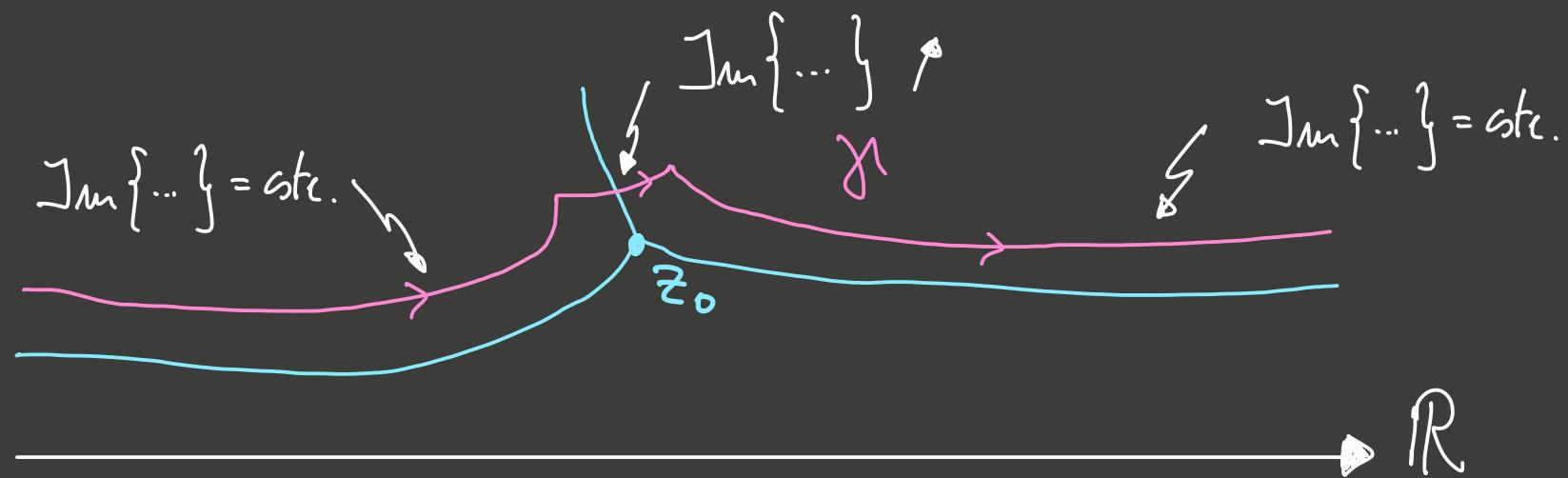
Remarks about dissipative paths:

If $\gamma : \mathbb{R} \ni t \mapsto \gamma(t) = \gamma_1(t) + i\gamma_2(t) \in S_\alpha$

$$\operatorname{Im} \int_0^z (e_1 - e_2)(s) ds \quad \xrightarrow{\text{along } \gamma} \Leftrightarrow \dot{\gamma}_1(t) \operatorname{Im}(e_1 - e_2)(\gamma(t)) + \dot{\gamma}_2(t) \operatorname{Re}(e_1 - e_2)(\gamma(t)) \geq 0$$



Stokes lines: $\sum_{z_0} = \left\{ z \in S_\alpha \text{ s.t. } \operatorname{Im} \left\{ \int_{z_0}^z (e_1 - e_2)(s) ds \right\} = 0 \right\}$



- $\exists \gamma$ diss. $\Leftrightarrow \sum_{z_0}$ as in the picture.

- If several crossing points z_0, z_1, \dots, z_d , \exists diss. path for at most one z_h