

Gran Sasso
Quantum Meeting
22-26 March 21

Adiabatic quantum dynamics and applications

Alain Joye
Institut Fourier
Univ. Grenoble Alpes

(Tentative) Program:

- 1) Adiabatic Theorems in Quantum Mechanics
 - "à la Kato"
 - superadiabatics
 - exponential
 - gapless
- 2) Applications to open quantum systems
 - Lindbladians
 - dephasing Lindbladians
 - Two-level system in a Bose field

C.

Two-level system:

$S_\alpha \ni z \mapsto H(z) \in M_2(\mathbb{C})$, analytic.

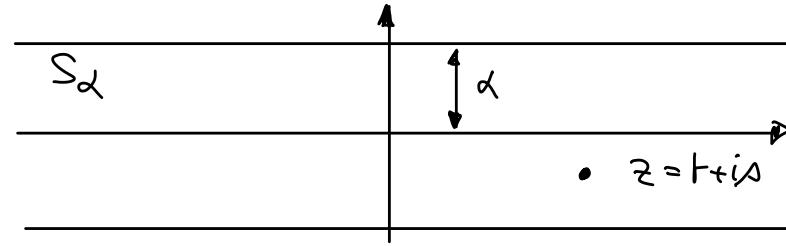
$\forall z \in S_\alpha, H^*(z) = H(\bar{z}).$

$\exists H(\pm\infty) = H(\pm\infty)^*$ s.t. $\sup_{|t| \leq \alpha} \|H(t+is) - H(\pm\infty)\| \leq f(t) \xrightarrow{t \rightarrow \pm\infty} 0$, $\int_{\mathbb{R}} f(t) dt < \infty$.

$\forall t \in \mathbb{R}, H(t) = \sum_j e_j(t) P_j(t)$,

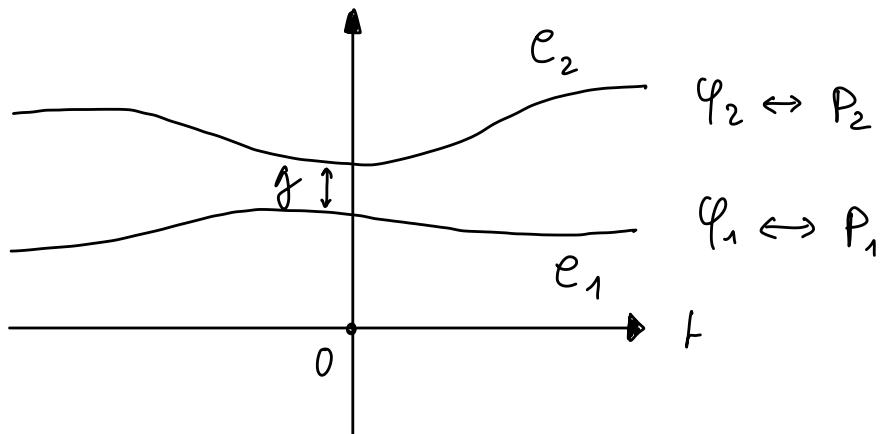
s.t. $\{e_j(t)\}_j = \sigma(H(t)) \subset \mathbb{R}$,

$$\inf_{t \in \mathbb{R}} e_2(t) - e_1(t) = g > 0.$$



$P_j(t) = |\psi_j(t)\rangle\langle\psi_j(t)|$; $\|\psi_j(t)\| = 1$; $\psi_j \in C^\infty(\mathbb{R})$.

$$\text{p.f. } \langle \varphi_j(t) | \varphi_j'(t) \rangle = 0$$



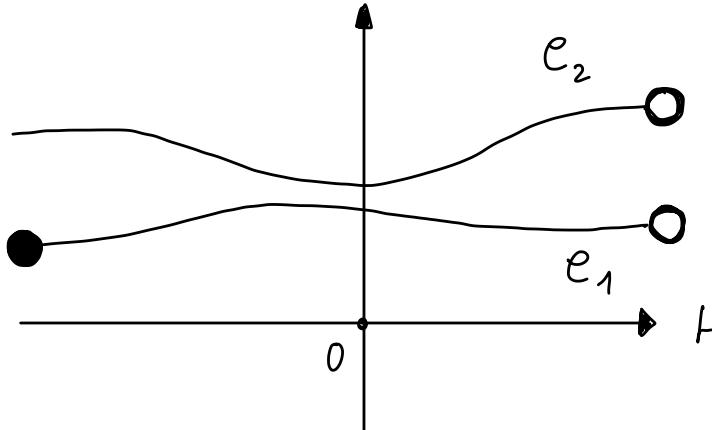
$\Leftrightarrow \varphi_j(t) = W(t) \varphi_j(0)$, where $W(t) = [P_1'(t), P_1(t)] W(t)$; $W(0) = \mathbb{1}$.

$$i\varepsilon \partial_t \Psi_\varepsilon = H(t) \Psi_\varepsilon, \quad t \in \mathbb{R}, \quad \|\Psi_\varepsilon(t)\| = 1.$$

$$\lim_{t \rightarrow -\infty} P_2(t) \Psi_\varepsilon(t) = 0$$

Q: $\lim_{t \rightarrow +\infty} P_2(t) \Psi_\varepsilon(t) = ?$

$$\Psi_\varepsilon(t) = \sum_{j=1}^2 C_j^\varepsilon(t) e^{-\frac{i}{\varepsilon} \int_0^t e_j(b) db} \varphi_j(t) \quad \text{o.f.} \quad \lim_{t \rightarrow -\infty} C_1^\varepsilon(t) = 1, \quad |C_1^\varepsilon(t)|^2 + |C_2^\varepsilon(t)|^2 = 1.$$



Q $\Leftrightarrow \lim_{t \rightarrow +\infty} C_2^\varepsilon(t) = ?$ // Transport

$$\begin{cases} C_j^\varepsilon'(t) = a_{jk}(t) e^{+\frac{i}{\varepsilon} \int_0^t (e_j(s) - e_k(s)) ds} C_k^\varepsilon(t); \quad C_j^\varepsilon(+\infty) = \delta_{j,1}, \quad j \neq k. \\ a_{jk}(t) = -\langle \varphi_j(t) | \varphi_k'(t) \rangle = -\langle \varphi_j(0) | \tilde{W}(t) [P_j'(t), P_k(t)] W(t) \varphi_k(0) \rangle = O(\ell(t)) \end{cases}$$

Note: $\lim_{t \rightarrow \pm\infty} C_j^\varepsilon(t) := C_j^\varepsilon(\pm\infty)$ exists since $\int_{\mathbb{R}} f(t) dt < \infty$.

Analytic continuation:

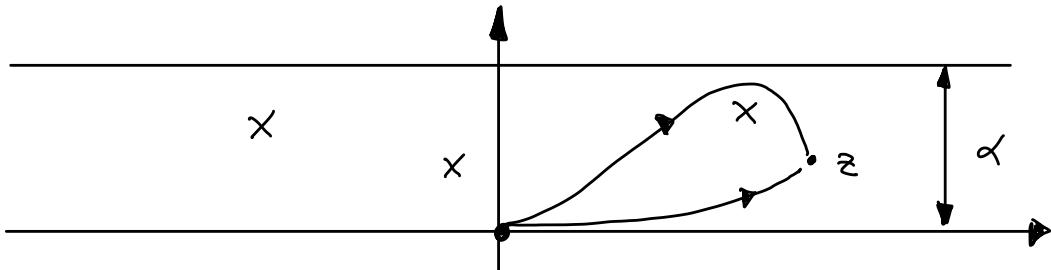
H is analytic in $S_d \Rightarrow \Psi_\epsilon$ has an analytic extension in S_d (sol. of ODE).

For $z \in S_d$, charact. eq. $\lambda^2 - \lambda \ln H(z) + \det H(z) = 0$ with analytic coeff.

$\Rightarrow c_j$ have analytic extensions in $S_d \setminus X$, where $X := \{ z \in S_d \text{ s.t. } e_1(z) = e_2(z) \}$.

e_j have at worse algebraic singularities at X (the set of cornering points.)

- X is finite (scatt. cond.),



- $X \cap \mathbb{R} = \emptyset$ (gap).

$\Rightarrow P_j$ have analytic extensions in $S_d \setminus X$, and W , hence φ_j also.

Note: These extensions depend on the way to reach $z \in S_d$ from 0. (Monodromy)

- c_j^ϵ admit analytic extensions from \mathbb{R} to $S_d \setminus X$ that we shall exploit.

Specifically for 2-level case:

$$z_0 \in X \Leftrightarrow (h_2 H(z_0))^2 - 4 \det H(z_0) = 0$$

$$\text{Generically: } e_2(z) - e_1(z) = \left\{ (h_2 H(z))^2 - 4 \det H(z) \right\}^{1/2} = c_{z_0} (z - z_0)^{1/2} (1 + O(z - z_0)).$$

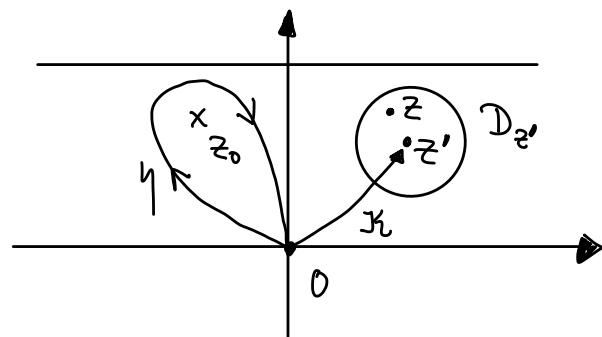
Also: $P_1(z), P_2(z)$ analytic in $S_d \setminus X$ with (generic) singularities $\sim (z - z_0)^{-1/2}$

Notations: Let $\gamma \subset S_d$ be a simple loop with base point at 0, negatively oriented. Let $z' \in S_d \setminus X$, $D_{z'} \subset S_d \setminus X$ be a neighborhood of z' .

H f analytic in $S_d \setminus X$, originally defined in D_0 ,

let $z \mapsto f(z)^\kappa$ its extension along κ in $D_{z'}$.

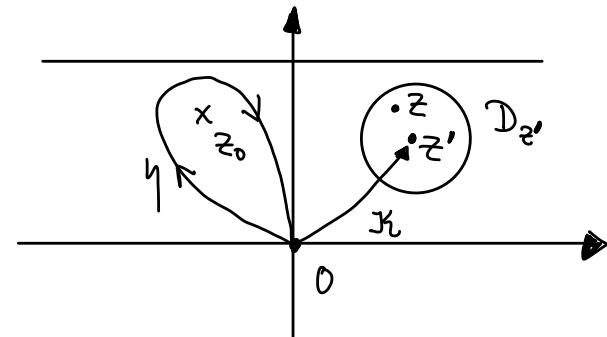
Set $z \mapsto f(z|\gamma)^\kappa$ the extension in $D_{z'}$ obtained along $\kappa \cdot \gamma$ (first γ , then κ)



Consequence: let $z_0 \in X$ be a simple zero of $z \mapsto (h_2 H z)^2 - 4 \det(Hz)$

$$j \neq k \in \{1, 2\}$$

- $e_j^{\kappa}(z|\eta) = e_k^{\kappa}(z)$, (exchange of eigenvalues)



- $\varphi_j^{\kappa}(z|\eta) = e^{-i\theta_j(\eta)} \varphi_k^{\kappa}(z)$, $\theta_j(\eta) \in \mathbb{C}$. (exchange of eigenvectors)

- $\Psi_{\epsilon}^{\kappa}(z) = \Psi_{\epsilon}(z)$ is indep. of κ , since $H(z)$ analytic in S_d .

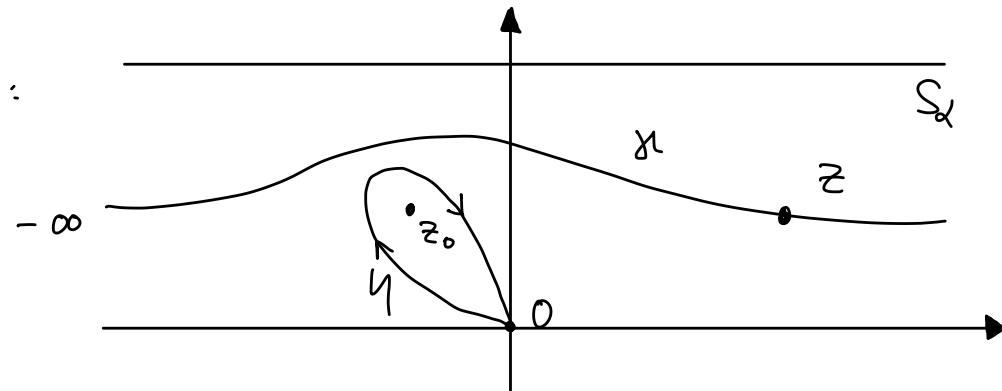
$$\Psi_{\epsilon}(z) = \sum_{j=1}^2 C_j^{\kappa}(z) e^{-\frac{i}{\epsilon} \left(\int_0^z e_j(s) ds \right)^{\kappa}} \varphi_j^{\kappa}(z) \equiv \sum_{j=1}^2 C_j^{\kappa}(z|\eta) e^{-\frac{i}{\epsilon} \left(\int_0^z e_j(s) ds \right)^{\kappa}} \varphi_j^{\kappa}(z|\eta)$$

Proposition: $C_2^{\kappa}(z) = e^{-i\theta_2(\eta)} e^{-\frac{i}{\epsilon} \int_{\eta}^z e_1(s) ds} C_1^{\kappa}(z|\eta) \quad \forall z \in D_{z'}$

Control of $C_1(+\infty|\eta)$ to get $C_2(+\infty)$:

$$C_1(-\infty) = 1$$

$$C_2(-\infty) = 0$$



$$\begin{cases} C_1(z) = 1 + \int_{-\infty}^z a_{12}(u) e^{+\frac{i}{\varepsilon} \int_0^u (e_1 - e_2)(s) ds} C_2(u) du \\ C_2(z) = \int_{-\infty}^z a_{21}(u) e^{+\frac{i}{\varepsilon} \int_0^u (e_2 - e_1)(s) ds} C_1(u) du \end{cases}$$

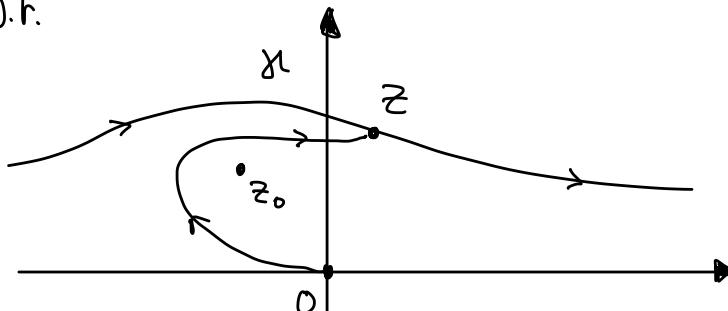
Rem: $\lim_{|t| \rightarrow \infty} C_j(t+is) = G_j(\pm\infty)$.
unif in $s \in [-\delta, \delta]$

Writing C_j the solution on \mathbb{R} , the monodromy relation yields

$$C_2(+\infty) = e^{-i\phi_1(\eta)} e^{-\frac{i}{\varepsilon} \int_\eta^\infty e_2(s) ds} C_1(+\infty)$$

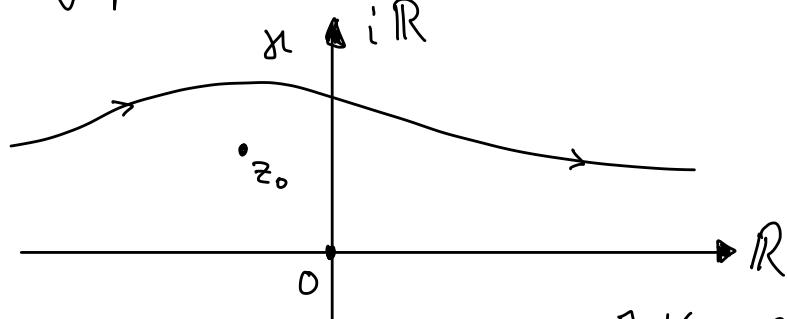
Derivative path: γ_1 , from $-\infty$ to $+\infty$, s.t.

$$\operatorname{Im} \int_0^z (e_1 - e_2)(\tau) d\tau \quad \text{along } \gamma_1$$



Theorem: Under the regularity and gap assumptions on $z \mapsto H(z)$, if

- $\exists \gamma$ a dissipative path
- $\exists! z_0 \in X$, generic s.t. :



J-Kunz-Pfister '91

Then: $i\varepsilon \partial_t \Psi_\varepsilon = H(t) \Psi_\varepsilon$, $t \in \mathbb{R}$, $\|\Psi_\varepsilon(t)\| = 1$, with $\lim_{t \rightarrow -\infty} P_2(t) \Psi_\varepsilon(t) = 0$

$$\Rightarrow \lim_{t \rightarrow +\infty} e^{+\frac{i}{\varepsilon} \int_0^t e_2(s) ds} \langle \varphi_2(t) | \Psi_\varepsilon(t) \rangle = e^{-i\phi_2(\eta)} e^{-\frac{i}{\varepsilon} \int_\eta^\infty e_1(s) ds} (1 + O(\varepsilon))$$

Noctily $\rightarrow | \cdot | \neq 1 \quad | \cdot | = e^{-\Gamma/\varepsilon}$

Rem: • This "Dykhne formula" yields Landau-Zener for small gaps.

• Landau-Zener : $\begin{pmatrix} t & \delta \\ \delta & -t \end{pmatrix} \rightsquigarrow \Psi_\varepsilon(t)$ via special facts.
 $\Rightarrow \lim_{t \rightarrow +\infty} \|P_2(t) \Psi_\varepsilon(t)\| = e^{-\frac{\pi}{4} \frac{\delta^2}{\varepsilon}}$, $\forall \delta, \varepsilon > 0$.

• Zener '32, Majorana '32, Landau '32, Dykhne '62, Hwang-Pechukas '77, Berry '90

Proof: It is enough to show that $C_1^x(+\infty) = 1 + O(\varepsilon)$.

$$\begin{aligned}
 C_1^x(z) &= 1 + \int_{-\infty}^z a_{12}(u) e^{+\frac{i}{\varepsilon} \int_0^u (e_1 - e_2)(s) ds} C_2^x(u) du \\
 C_2^x(z) &= \int_{-\infty}^z a_{21}(u) e^{+\frac{i}{\varepsilon} \int_0^u (e_2 - e_1)(s) ds} C_1^x(u) du \\
 &\quad \underbrace{\frac{\varepsilon}{i} \frac{1}{(e_2 - e_1)(u)} \frac{d}{du} \left(e^{\frac{i}{\varepsilon} \int_0^u (e_2 - e_1)(s) ds} \right)}_{\text{from } \frac{d}{du} C_1^x(u)} \\
 &= \frac{\varepsilon}{i} \frac{a_{21}(z)}{(e_2 - e_1)(z)} e^{+\frac{i}{\varepsilon} \int_0^z (e_2 - e_1)(s) ds} C_1^x(z) - 0 - \frac{\varepsilon}{i} \int_{-\infty}^z \frac{d}{du} \left(\frac{a_{21}(u)}{(e_2 - e_1)(u)} \right) e^{+\frac{i}{\varepsilon} \int_0^u (e_2 - e_1)(s) ds} C_1^x(u) du \\
 &\quad - \frac{\varepsilon}{i} \int_{-\infty}^z \frac{a_{21}(u)}{(e_2 - e_1)(u)} a_{12}(u) C_2^x(u) du \quad \xrightarrow{\text{from } \frac{d}{du} C_1^x(u)}
 \end{aligned}$$

New variables: $X_1(z) := C_1^x(z)$; $X_2(z) := e^{+\frac{i}{\varepsilon} \int_0^z (e_1 - e_2)(s) ds} C_2^x(z)$ so that

$$\begin{aligned}
 X_1(z) &= 1 + \int_{-\infty}^z a_{12}(u) X_2(u) du \\
 X_2(z) &= \varepsilon \left\{ A(z) X_1(z) + \int_{-\infty}^z B(u) e^{\frac{i}{\varepsilon} \int_u^z (e_1 - e_2)(s) ds} X_1(u) du + \int_{-\infty}^z C(u) e^{\frac{i}{\varepsilon} \int_u^z (e_1 - e_2)(s) ds} X_2(u) du \right\}
 \end{aligned}$$

where

Scatt. hypoth: $\exists c < \infty$ s.t. $\max \left\{ |A(z)|, \int_{-\infty}^z |\beta(u)| du, \int_{-\infty}^z |\gamma(u)| du, \int_{-\infty}^z |\alpha_{12}(u)| du \right\} \leq c$

by definition: $\left| e^{\frac{i}{\varepsilon} \int_u^z (\epsilon_1 - \epsilon_2)(s) ds} \right| = e^{-\frac{1}{\varepsilon} \operatorname{Im} \int_u^z (\epsilon_1 - \epsilon_2)(s) ds} \leq 1 \quad \forall \varepsilon > 0.$

$$\Rightarrow |X_1(z)| \leq 1 + c \sup_{u \in \mathcal{X}} |X_2(u)|$$

$$|X_2(z)| \leq \varepsilon c \left(\sup_{u \in \mathcal{X}} |X_1(u)| + \sup_{u \in \mathcal{X}} |X_2(u)| \right)$$

and $\sup_{u \in \mathcal{X}} |X_1(u)| \leq 1 + c \sup_{u \in \mathcal{X}} |X_2(u)|$

$$\sup_{u \in \mathcal{X}} |X_2(u)| \leq \varepsilon c \left(1 + (c+1) \sup_{u \in \mathcal{X}} |X_2(u)| \right).$$

constant.
↓

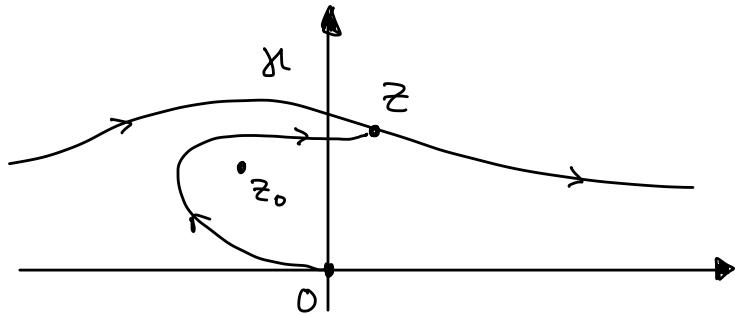
Thus $\sup_{u \in \mathcal{X}} |X_2(u)| \left(1 - \varepsilon c(c+1) \right) \stackrel{\varepsilon \text{ small}}{\leq} \varepsilon c \Rightarrow \sup_{u \in \mathcal{X}} |X_2(u)| \leq \varepsilon \tilde{c}$.

Finally: $|X_1(z) - 1| \leq c \sup_{u \in \mathcal{X}} |X_2(u)| \leq \varepsilon c \tilde{c}$, unif in $z \in \mathcal{X}$.



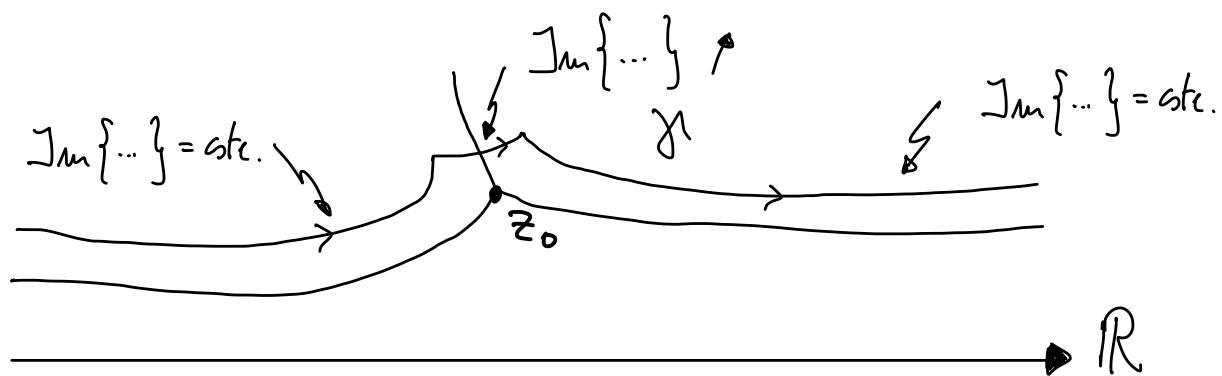
Remarks about dissipative paths:

If $\gamma: \mathbb{R} \ni t \mapsto \gamma(t) = \gamma_1(t) + i\gamma_2(t) \in S_\alpha$



$$\operatorname{Im} \int_0^z (e_1 - e_2)(s) ds \quad \text{along } \gamma \iff \dot{\gamma}_1(t) \operatorname{Im}(e_1 - e_2)(\gamma(t)) + \dot{\gamma}_2(t) \operatorname{Re}(e_1 - e_2)(\gamma(t)) \geq 0$$

Stokes lines: $\sum_{z_0} = \left\{ z \in S_\alpha \text{ s.t. } \operatorname{Im} \left\{ \int_{z_0}^z (e_1 - e_2)(s) ds \right\} = 0 \right\}$



- $\exists \gamma \text{ diss.} \Leftrightarrow \sum_{z_0}$ as on the picture.
- If several cornering points z_0, z_1, \dots, z_d , \exists diss. path for at most one z_h