

Gran Sasso
Quantum Meeting
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Adiabatic quantum dynamics and applications

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(Tentative) Program:

1) Adiabatic Theorems in Quantum Mechanics

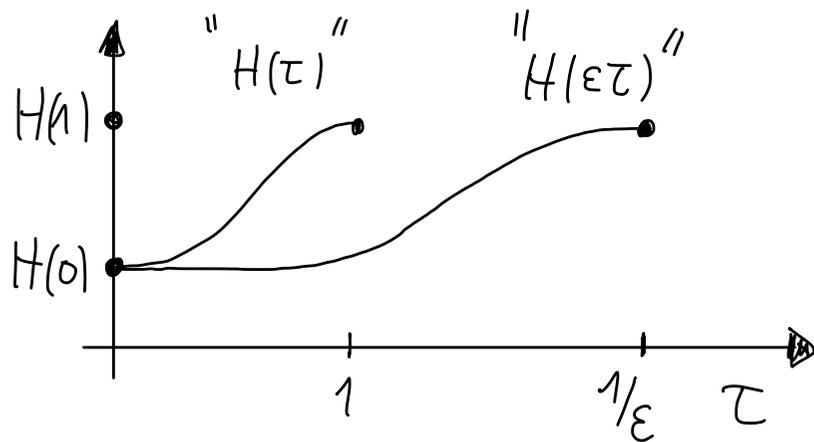
- "à la Kato"
- superadiabatic
- exponential
- gapless

2) Applications to open quantum systems

- Lindbladians
- dephasing Lindbladians
- Two-level system in a Bose field

Adiabatic Theorems in Quantum Mechanics

Context: approximate solution to the time-dependent Schrödinger equation with slowly varying time-dependent Hamiltonian



$H: [0, 1] \rightarrow \mathcal{L}(\mathcal{H})$, \mathcal{H} separable Hilbert space
 $\tau \mapsto H(\tau) = H^*(\tau)$; smooth

Determine $\chi_\varepsilon(\tau) \in \mathcal{H}$ s.t. $\tau \in [0, 1/\varepsilon]$

$$\begin{cases} i \partial_\tau \chi_\varepsilon = H(\varepsilon\tau) \chi_\varepsilon ; \tau \in (0, 1/\varepsilon) \\ \chi_\varepsilon(0) = \varphi_\varepsilon ; \text{ as } \varepsilon \rightarrow 0. \end{cases}$$

Change of variable: $t := \varepsilon\tau \in [0, 1]$; $\Psi_\varepsilon(t) := \chi_\varepsilon(t/\varepsilon)$

$$\begin{cases} i \varepsilon \partial_t \Psi_\varepsilon = H(t) \Psi_\varepsilon ; t \in (0, 1) \\ \Psi_\varepsilon(0) = \varphi_\varepsilon ; \text{ as } \varepsilon \rightarrow 0. \end{cases} \Leftrightarrow$$

Determine the propagator $U_\varepsilon: [0, 1] \times [0, 1] \rightarrow \mathcal{L}(\mathcal{H})$

$$\text{s.t. } \begin{cases} i \varepsilon \partial_t U_\varepsilon(t, t_0) = H(t) U_\varepsilon(t, t_0) \\ U_\varepsilon(t_0, t_0) = \mathbb{1} ; \text{ as } \varepsilon \rightarrow 0. \end{cases}$$

Remark:

For $H(t) \in C^2([0,1]; \mathcal{L}(\mathcal{H}))$, existence and uniqueness thanks to Dyson series.

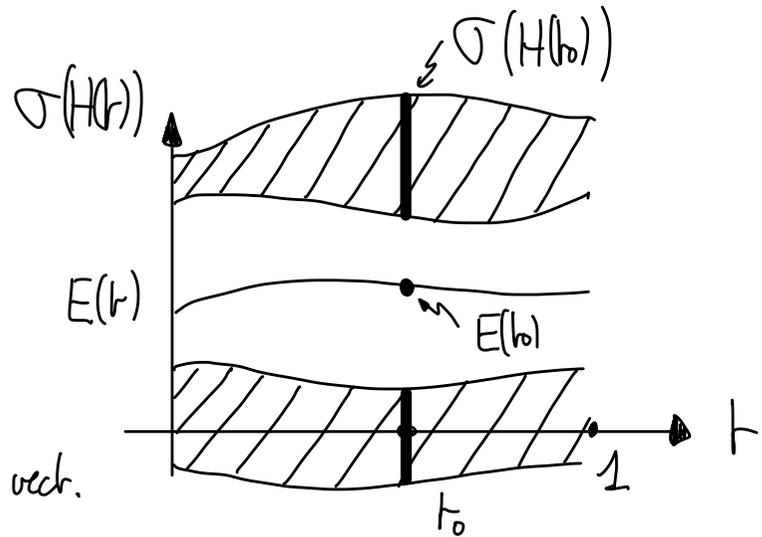
Slogan: "Once in an eigenstate, always in an eigenstate":

Let $H(t) = H^*(t)$ o.k.

$E(t) \in \sigma(H(t))$ isolated simple eigenvalue

$\varphi(t) \in \mathcal{H}$ $H(t)\varphi(t) = E(t)\varphi(t)$; $\|\varphi(t)\| = 1$

and $\langle \varphi(t) | \varphi'(t) \rangle \equiv 0$, instantaneous eigenvech.



Let $t_0 \in [0,1]$, $\Psi_\varepsilon(t, t_0) := U_\varepsilon(t, t_0)\varphi(t_0)$ o.k.
$$\begin{cases} i\varepsilon \partial_t \Psi_\varepsilon(t, t_0) = H(t)\Psi_\varepsilon(t, t_0) \\ \Psi_\varepsilon(t_0, t_0) = \varphi(t_0) \end{cases}$$

Then,

$$\Psi_\varepsilon(t, t_0) = e^{-\frac{i}{\varepsilon} \int_{t_0}^t E(s) ds} \varphi(t) + O(\varepsilon) \quad \forall t, t_0 \in [0,1]$$

Bam-Fuchs '28
Kato '50

Note: We will often take $t_0 = 0$ and write $U_\varepsilon(t) := U_\varepsilon(t, 0)$, o.k. $U_\varepsilon(0) = \mathbb{1}$.

Upgrade: Let \mathcal{H} be a separable Hilbert space. Assume:

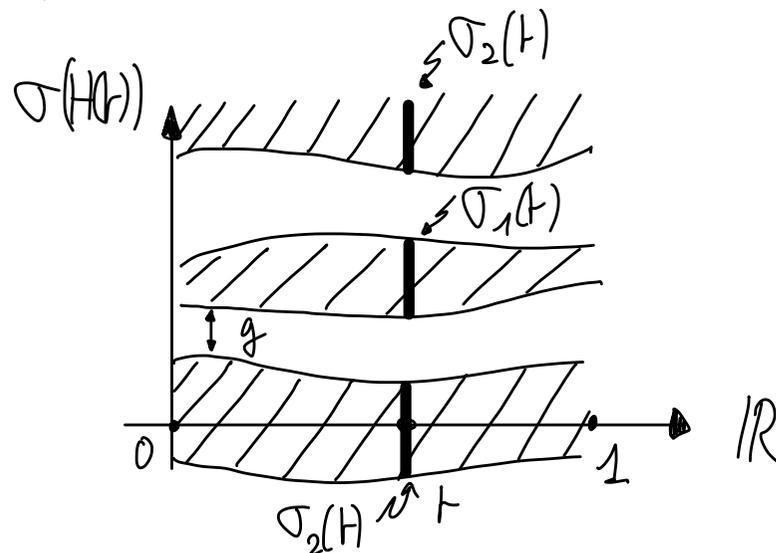
(R) $[0,1] \ni t \mapsto H(t) = H^*(t) \in \mathcal{L}(\mathcal{H})$ (or defined on $\mathcal{D} \subset \mathcal{H}$ indep. of t)

H strongly C^2 on $[0,1]$; $\exists \ell \in \mathbb{R}$ s.t. $H(t) \geq \ell$, $\forall t \in [0,1]$.

(G) $\sigma(H(t)) = \sigma_1(t) \cup \sigma_2(t)$ s.t.

$\sigma_1(t)$ bounded, $\exists g > 0$ s.t.

$\inf_{t \in [0,1]} \text{dist}(\sigma_1(t), \sigma_2(t)) \geq g > 0$



Riesz Projection:

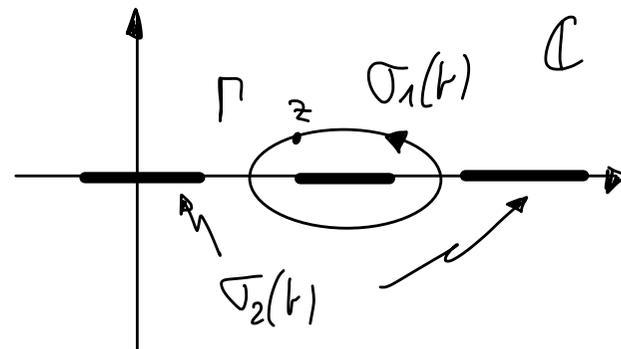
Let $P(t) := -\frac{1}{2i\pi} \int_{\Gamma} (H(t) - z)^{-1} dz \iff \sigma_1(t)$

$P^\perp(t) := \mathbb{1} - P(t) \iff \sigma_2(t)$

s.t. $P(t) = P^2(t) = P^*(t)$; $t \mapsto P(t)$ str- $C^2([0,1])$

$[H(t), P(t)] = 0$; $(P(t): \mathcal{H} \rightarrow \mathcal{D})$

$\sigma(H(t)|_{P(t)\mathcal{H}}) = \sigma_1(t)$, $\sigma(H(t)|_{P^\perp(t)\mathcal{H}}) = \sigma_2(t)$.



Intertwining or \parallel -transport or Kato operator:

Let $[0, 1] \ni t \rightarrow P(t) = P^2(t) \in \mathcal{L}(\mathcal{B})$; \mathcal{B} a Banach, be strongly C^1

Consider $W(t) \in \mathcal{L}(\mathcal{B})$ the solution to
$$\begin{cases} W'(t) = [P'(t), P(t)] W(t) \\ W(0) = \mathbb{1} \end{cases} \quad (\text{with } ' \Leftrightarrow \partial_t)$$

Lemma: $W(t)P(0) = P(t)W(t), \quad \forall t \in [0, 1].$ Kato '50

Rem: On a Hilbert, if $P(t) = P^*(t) \Rightarrow W(t)$ unitary

In any case $W^{-1}(t) \in \mathcal{L}(\mathcal{B})$ exists and
$$\begin{cases} W^{-1}(t)' = -W^{-1}(t) [P'(t), P(t)] \\ W^{-1}(0) = \mathbb{1}. \end{cases}$$

If $\varphi = P(0)\varphi \Rightarrow W(t)\varphi = P(t)W(t)\varphi.$

On a Hilbert, if $P(t) = P^*(t)$ has rank 1 $\Rightarrow P(0) = |\varphi\rangle\langle\varphi|$ and $\varphi(t) := W(t)\varphi$

$\Rightarrow \|\varphi(t)\| \equiv 1, \langle\varphi(t)|\varphi'(t)\rangle \equiv 0$ and $P(t) = |\varphi(t)\rangle\langle\varphi(t)|$

Proof: $W(t)P(0)$ and $P(t)W(t)$ satisfy the same linear diff. eq. with initial condition $P(0)$:

(Writing $A \equiv A(t)$; $A(0) = A(0)$)

- $P^2 = P \Rightarrow P'P + PP' = P' \Rightarrow PP'P \equiv 0.$
- $(W P(0))' = [P', P] W P(0)$
- $(PW)' = P(P'P - PP')W + P'W = -PP'W + (P'P + PP')W = P'P(PW) = [P', P](PW)$

Finally: if $P = P^* = P^2$, and $\varphi_0 = P(0)\varphi_0$, for $\varphi(t) := W(t)\varphi_0$ we have

$$\langle \varphi | \varphi' \rangle = \langle P\varphi | [P', P]\varphi \rangle = \langle \varphi | P[P', P]P\varphi \rangle \equiv 0. \quad \square$$

Dynamical phase operator: Φ_ε bdd on \mathcal{B} , a Banach space, defined by

$$i\varepsilon \Phi_\varepsilon'(t) = (W'(t)H(t)W(t))\Phi_\varepsilon(t) \quad \& \quad \Phi_\varepsilon(0) = \mathbb{1}, \quad \text{where } [H(t), P(t)] \equiv 0.$$

Lemma: $[\Phi_\varepsilon(t), P(0)] = 0 \quad \forall t \in [0, 1].$ (Evolution within $P(0)\mathcal{B}$.)

Proof: $P(0)\Phi_\varepsilon(t)$ and $\Phi_\varepsilon(t)P(0)$ satisfy the same lin. diff. eq. with initial cond. $P(0)$.

Rem: In a Hilbert space with $H(t) = H^*(t)$ $\Phi_\varepsilon(t)$ is unitary.

If, moreover, $P(t) \leftrightarrow \mathcal{V}_1(t) := \{E(t)\} \Rightarrow \Phi_\varepsilon(t)|_{P(0)\mathcal{H}} = e^{-\frac{i}{\varepsilon} \int_0^t E(s) ds} P(0)$.

Adiabatic evolution:

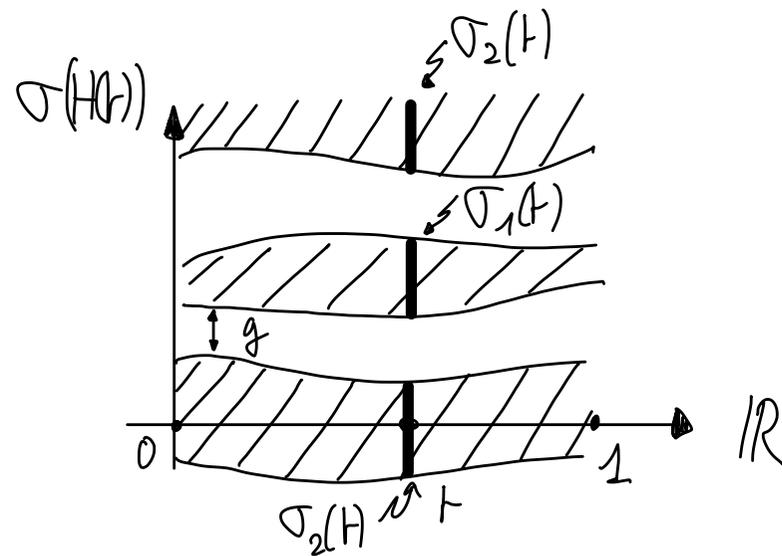
We now define $V_\varepsilon(t) := W(t)\Phi_\varepsilon(t)$. Then $V_\varepsilon(t)P(0) = P(t)V_\varepsilon(t)$.

Moreover: $i\varepsilon V_\varepsilon(t)' = (H(t) + i\varepsilon [P'(t), P(t)])V_\varepsilon(t) \quad \& \quad V_\varepsilon(0) = \mathbb{1}.$

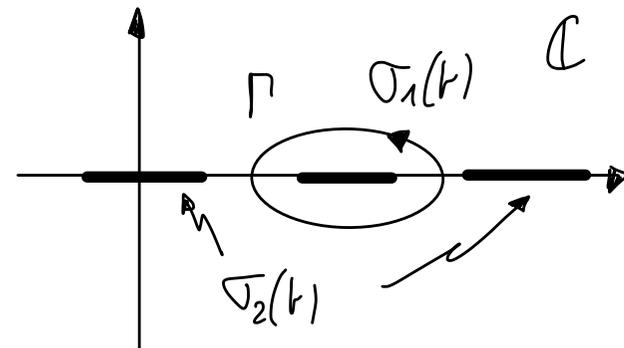
Reminder: \mathcal{H} is a separable Hilbert space.

(R) $[0,1] \ni t \mapsto H(t) = H^*(t) \in \mathcal{L}(\mathcal{H})$ (or defined on $\mathcal{D} \subset \mathcal{H}$ indep. of t)
 H strongly C^2 on $[0,1]$; $\exists \epsilon \in \mathbb{R}$ s.t. $H(t) \geq \epsilon$, $\forall t \in [0,1]$.

(G) $\sigma(H(t)) = \sigma_1(t) \cup \sigma_2(t)$ s.t.
 $\sigma_1(t)$ bounded, $\exists g > 0$ s.t.
 $\inf_{t \in [0,1]} \text{dist}(\sigma_1(t), \sigma_2(t)) \geq g > 0$



Let $P(t) := -\frac{1}{2i\pi} \int_{\Gamma} (H(t) - z)^{-1} dz \iff \sigma_1(t)$



Adiabatic Theorem: Assume the above. (Nenciu '80, Avron-Seiler-Yaffe '87)

Let $U_\varepsilon(t)$ s.t. $i\varepsilon U'_\varepsilon(t) = H(t)U_\varepsilon(t)$ & $U_\varepsilon(0) = \mathbb{1}$, and $V_\varepsilon(t) = W(t)\Phi_\varepsilon(t)$.

Then: $\exists C > 0$ s.t. $\forall t \in [0, 1]$, $\forall \varepsilon > 0$,

$$\|U_\varepsilon(t) - V_\varepsilon(t)\| \leq C(\varepsilon t). \quad (\text{with } U_\varepsilon(t), W(t), \Phi_\varepsilon(t) \text{ unitary})$$

Remarks: • If $H(t)$ defined on $I \supset [0, 1]$, the estimate above holds unif. in I .

• The transition amplitudes between spectral subspaces vanish:

$$\|(\mathbb{1} - P(t))U_\varepsilon(t)P(0)\| = O(\varepsilon t) \Rightarrow \text{transition probabilities are of order } \varepsilon^2. \quad (0 \leq t \leq 1)$$

• Reduces to the case of Kato in case $\text{rank } P(t) = 1$ & $\mathcal{D}_1(t) = \{E(t)\}$

• Estimate on the transition amplitudes between spectral subspaces of H is sharp.

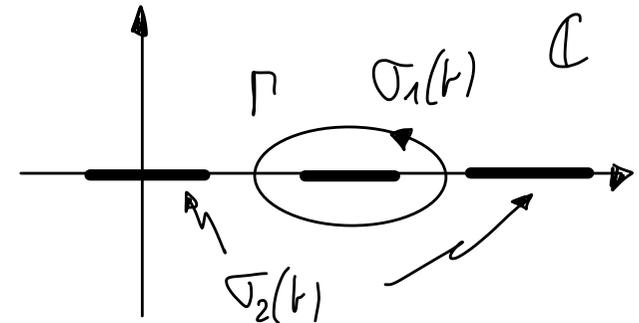
• Estimate on the evolution op. can (and will) be improved.

Proof: For short, write A for $A(t)$, $A(0)$ for $A(0)$.

Define $\Omega_\varepsilon(t)$ by $U_\varepsilon(t) = V_\varepsilon(t) \Omega_\varepsilon(t) \Leftrightarrow \Omega_\varepsilon(t) = V_\varepsilon^{-1}(t) U_\varepsilon(t)$.

$$i\varepsilon \Omega_\varepsilon' = -V_\varepsilon^{-1} (H + i\varepsilon [P', P]) U_\varepsilon + V_\varepsilon^{-1} H U_\varepsilon = -i\varepsilon V_\varepsilon^{-1} [P', P] U_\varepsilon \quad ; \quad \Omega_\varepsilon(0) = \mathbb{1}.$$

i.e:
$$\Omega_\varepsilon(t) = \mathbb{1} + \int_0^t V_\varepsilon^{-1} [P', P] U_\varepsilon ds$$



Integration by parts: let $B \in \mathcal{L}(\mathcal{H})$ and set

$$\mathcal{R}(B) := -\frac{1}{2i\pi} \int_{\Gamma} (H-z)^{-1} B (H-z)^{-1} dz : \mathcal{H} \rightarrow \mathcal{D}. \quad \text{Then: } \underline{[\mathcal{R}(B), H] = [P, B]}.$$

$$\text{Indeed: } -\frac{1}{2i\pi} \int_{\Gamma} [(H-z)^{-1} B (H-z)^{-1}, H] dz \stackrel{H \mapsto H-z}{=} -\frac{1}{2i\pi} \int_{\Gamma} [(H-z)^{-1}, B] dz.$$

For $B = P'$, $[\mathcal{R}(P'), H] = [P, P']$, where $\rho \mapsto \mathcal{R}(P')(\rho) \in \text{dom-}C^1$.

$$\begin{aligned} \Rightarrow \underline{\Omega_\varepsilon(t) - \mathbb{1}} &= \int_0^t V_\varepsilon^{-1} [\mathcal{R}(P'), H] U_\varepsilon ds = \int_0^t i\varepsilon \frac{d}{ds} \left\{ V_\varepsilon^{-1} \mathcal{R}(P') U_\varepsilon \right\} ds \\ &+ \int_0^t \left\{ i\varepsilon V_\varepsilon^{-1} [P', P] \mathcal{R}(P') U_\varepsilon - i\varepsilon V_\varepsilon^{-1} (\mathcal{R}(P'))' U_\varepsilon \right\} ds \stackrel{\text{unitarity}}{=} \underline{O(\varepsilon t)}. \quad \square \end{aligned}$$

Remarks

- Besides regularity in t , existence of the propagators involved, separation of $\sigma(H(t))$, no need for self adjointness of $H(t)$ to get the integral expression for $\Omega_\varepsilon(t) - \Pi$.
 - To conclude, it is enough to have $\sup_{t \in [0,1]} \|V_\varepsilon^{\pm 1}(t)\| = v < \infty \Leftrightarrow \sup_{\substack{t \in [0,1] \\ \varepsilon > 0}} \|\phi_\varepsilon^{\pm 1}(t)\| < \infty$
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$$\text{Indeed } \|\Omega_\varepsilon(t) - \Pi\| \leq c \cdot v \varepsilon \sup_{t \in [0,1]} \|U_\varepsilon(t)\|$$

$$\Rightarrow \|U_\varepsilon(t)\| = \|V_\varepsilon(t) \Omega_\varepsilon(t)\| \leq v + c v^2 \varepsilon \sup_{s \in [0,1]} \|U_\varepsilon(s)\|$$

$$\text{i.e. } \sup_{s \in [0,1]} \|U_\varepsilon(s)\| (1 - c v^2 \varepsilon) \leq v \Rightarrow \sup_{s \in [0,1]} \|U_\varepsilon(s)\| \leq 2v \quad \forall \varepsilon \leq \frac{1}{2c v^2}.$$

$$\text{and eventually } \Omega_\varepsilon(t) - \Pi = O(\varepsilon).$$