

Grenoble
Quantum Meeting
22-26 March 21

Adiabatic quantum dynamics and applications

Alain Joye
Institut Fourier
Univ. Grenoble Alps

(Tentative) Program :

1) Adiabatic Theorems in Quantum Mechanics

- "à la Kato"
- superadiabatic
- exponential
- gapless

2) Applications to open quantum systems

- Lindbladians
- dephasing Lindbladians
- Two-level system in a Box field

Superadiabatics "Systematic improvement of the approximation"

- Lennard '59, Sandro-Gamido '64, Beny '84, Nenciu '93, J.-Pfister '93

Idea. Heisenberg eq. for the projector:

$$Q_\varepsilon(t) = U_\varepsilon^{-1}(t) P(0) U_\varepsilon(t) \quad \text{s.t.} \quad i\varepsilon Q'_\varepsilon(t) = [H(t), Q_\varepsilon(t)]; \quad Q_\varepsilon(0) = P(0).$$

$$\text{Observe: } Q'_\varepsilon = [[Q'_\varepsilon, Q_\varepsilon], Q_\varepsilon] \quad \left(= Q'_\varepsilon Q_\varepsilon - Q_\varepsilon Q'_\varepsilon Q_\varepsilon - Q_\varepsilon Q'_\varepsilon Q_\varepsilon + Q_\varepsilon Q'_\varepsilon = Q'_\varepsilon Q_\varepsilon + Q_\varepsilon Q'_\varepsilon \right)$$

$$\text{Heisenberg} \Leftrightarrow [H(t) - i\varepsilon [Q'_\varepsilon, Q_\varepsilon], Q_\varepsilon] = 0.$$

Formally:

$$\text{Order 0: } Q_\varepsilon(t) := P(t) + \varepsilon R_0(t, \varepsilon) \quad (\text{hoping } R_0(t, \varepsilon), R'_0(t, \varepsilon) \text{ are } O(1) \text{ as } \varepsilon \rightarrow 0)$$

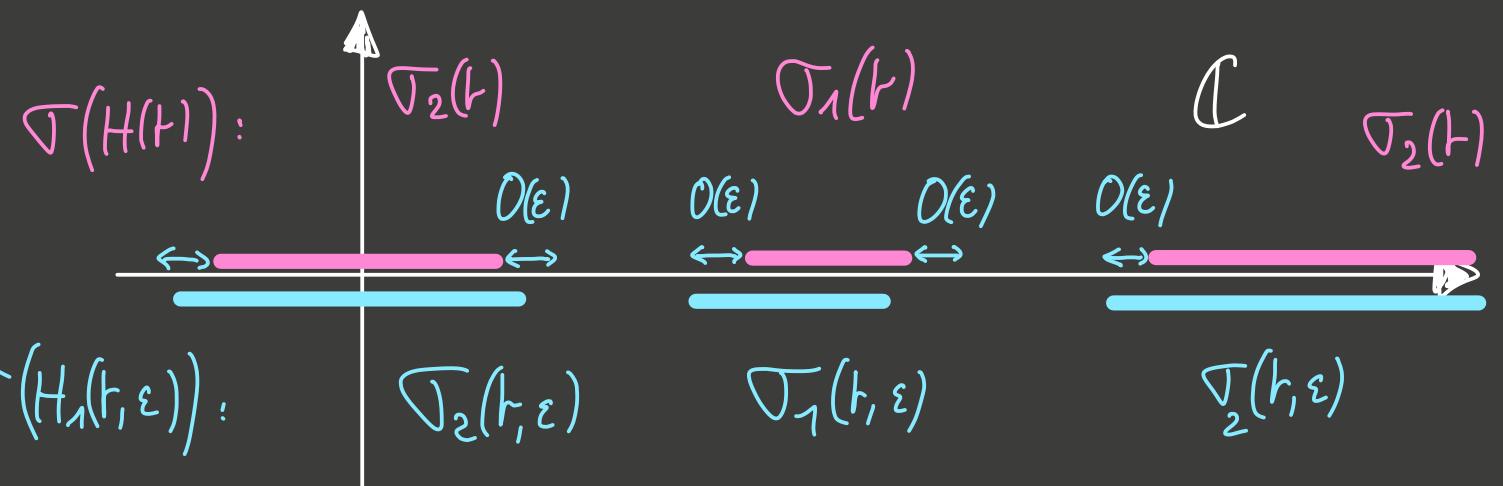
$$\Rightarrow \text{leading order } [H(t) - i\varepsilon [P(t), P(t)], Q_\varepsilon(t)] = O(\varepsilon^2)$$

$$\text{Order 1: } Q_\varepsilon(t) := Q_1(t, \varepsilon) + \varepsilon^2 R_1(t, \varepsilon) \quad \text{n.b.}$$

$$Q_1(t, \varepsilon) \leftrightarrow H_1(t, \varepsilon) = H(t) - i\varepsilon [P(t), P(t)] \quad (\text{hoping } R_1(t, \varepsilon), R'_1(t, \varepsilon) \text{ are } O(1) \text{ as } \varepsilon \rightarrow 0)$$

$$\text{Perturbation theory: } H_1(t, \varepsilon) = H(t) - i\varepsilon [P(t), P(t)]$$

(ε small)



$$T(H_1(t, \varepsilon)) = T_1(t, \varepsilon) \cup T_2(t, \varepsilon), \quad T_1(t, \varepsilon) \text{ bdd}, \quad \text{dist}(T_1(t, \varepsilon) \cup T_2(t, \varepsilon)) \geq g - O(\varepsilon)$$

Thus $Q_1(t, \varepsilon) := -\frac{1}{2i\pi} \int_{\Gamma} (H_1(t, \varepsilon) - z)^{-1} dz$ well defined spectral proj. of $H_1(t, \varepsilon)$

$$= P(t) + O(\varepsilon) \quad \left(\text{of class } C^{k-1} \text{ if } H \text{ of class } C^k. \right)$$

Leading order: $[H(t) - i\varepsilon [Q_1(t, \varepsilon), Q_1(t, \varepsilon)], Q_\varepsilon(t)] = O(\varepsilon^3) \quad \left(\text{if } R_1^{(1)}(t, \varepsilon) = O(1) \right)$

Order 2: $Q_\varepsilon(t) := Q_2(t, \varepsilon) + \varepsilon^3 R_2(t, \varepsilon) \quad \text{o.t.}$

$$Q_2(t, \varepsilon) \longleftrightarrow H_2(t, \varepsilon) = H(t) - i\varepsilon [Q_1(t, \varepsilon), Q_1(t, \varepsilon)] \quad \left(\text{hoping } R_2^{(1)}(t, \varepsilon) = O(1) \right)$$

well defined by perturbation theory and $Q_2(t, \varepsilon) = P(t) + O(\varepsilon)$.

Proposition: Let $[0,1] \ni t \mapsto H(t)$ be C^∞ and suppose assumption (G) holds.

Set $H_0(t, \varepsilon) := H(t)$; $Q_0(t, \varepsilon) := P(t)$; $K_0(t, \varepsilon) = i [Q'_0(t, \varepsilon), Q_0(t, \varepsilon)]$.

$\forall n \in \mathbb{N}$, $\exists c_n < \infty$, $\varepsilon_n > 0$ s.t. $\forall \varepsilon < \varepsilon_n \quad \forall 1 \leq q \leq n$, the operators

$H_q(t, \varepsilon) = H(t) - \varepsilon K_{q-1}(t, \varepsilon)$ \longleftrightarrow $Q_q(t, \varepsilon) \longleftrightarrow K_q(t, \varepsilon)$ are well defined C^∞ , and

$$\|K_q(t, \varepsilon) - K_{q-1}(t, \varepsilon)\| \leq c_n \varepsilon^q \quad (\text{as well as all derivatives})$$

Quasi proof: By induction. True for $q=1$, since $Q_1(\varepsilon) = Q_0(\varepsilon) + O(\varepsilon)$.

Assume the result for n .

$$\| \cdot \| \leq c_n \varepsilon^n \quad \text{by induction (+ derivatives)}$$

$$C^\infty \ni H_{n+1}(\varepsilon) := H - \varepsilon K_n(\varepsilon) = H_n(\varepsilon) - \varepsilon \overbrace{(K_n(\varepsilon) - K_{n-1}(\varepsilon))}^{\| \cdot \| \leq c_n \varepsilon^n}$$

Perturbation theory, for ε small enough, $Q_{n+1}(\varepsilon) = Q_n(\varepsilon) + O(\varepsilon^{n+1}) \quad (+ \text{ derivatives})$

$$\Rightarrow K_{n+1}(\varepsilon) = i [Q'_{n+1}(\varepsilon), Q_{n+1}(\varepsilon)] = K_n(\varepsilon) + O(\varepsilon^{n+1}) \quad (+ \text{ derivatives})$$

□

Remarks:

- No need of self-adjointness; just separation of spectrum & perturbation theory
- If $\begin{pmatrix} d \\ dt \end{pmatrix}^j H(t_0) = 0 \quad \forall j \leq m \Rightarrow H_m(t_0, \varepsilon) = H(t_0), \quad Q_m(t_0, \varepsilon) = P(t_0), \quad \forall m \leq m$

At step m Let

$$\begin{aligned} W_m(t, \varepsilon) &\leftrightarrow i [Q'_m(t, \varepsilon), Q_m(t, \varepsilon)] \equiv K_m(t, \varepsilon) \\ \phi_m(t, \varepsilon) &\leftrightarrow \frac{i}{\varepsilon} \bar{W}_m^{-1}(t, \varepsilon) H_m(t, \varepsilon) W_m(t, \varepsilon) \\ V_m(t, \varepsilon) &\leftrightarrow \frac{i}{\varepsilon} (H_m(t, \varepsilon) + \varepsilon K_m(t, \varepsilon)) \end{aligned}$$

s.t. $V_m(t, \varepsilon) Q_m(0, \varepsilon) = Q_m(t, \varepsilon) V_m(t, \varepsilon)$

Define $S_m(t, \varepsilon)$ by $U_\varepsilon(t) := V_m(t, \varepsilon) S_m(t, \varepsilon)$ so that

$$S_m(t, \varepsilon) = 1 \mathbb{I} + i \int_0^t V_m^{-1}(s, \varepsilon) \underbrace{\left(K_m(s, \varepsilon) - K_{m-1}(s, \varepsilon) \right)}_{O(\varepsilon^m)} U_\varepsilon(s) ds$$

Theorem: Under the standing assumptions $(\mathbb{J} \text{- Pfister '93})$

$$\| U_\varepsilon(t) - V_m(t, \varepsilon) \| \leq C_m(\varepsilon^m t)$$

$$\| (\mathbb{I} - Q_m(t, \varepsilon)) U_\varepsilon(t) Q_m(0, \varepsilon) \| \leq C_m(\varepsilon^m t)$$

Rem: • Integration by parts $\Rightarrow \varepsilon^m \mapsto \varepsilon^{m+1}$

- $H_m(t, \varepsilon) = H(t) + \varepsilon H^{(1)}(t) + \varepsilon^2 H^{(2)}(t) + \varepsilon^3 H^{(3)}(t) + \dots$
- $Q_m(t, \varepsilon) = P(t) + O(\varepsilon)$
- If $\left(\frac{d}{dt}\right)^j H(0) = 0 = \left(\frac{d}{dt}\right)^j H(1) \quad \forall j \leq m \quad \Rightarrow \| (\mathbb{I} - P(1)) U_\varepsilon(1) P(0) \| = O(\varepsilon^m)$
- If $\text{Rank } P(t) = 1, \quad \mathcal{O}_1(t) = \{E(t)\}, \quad P(0) = 1 \times 1,$
 $\exists \quad E_m(t, \varepsilon) = E(t) + \varepsilon^2 E^{(2)}(t) + \varepsilon^3 E^{(3)}(t) + \dots \in \mathcal{O}(H_m(t, \varepsilon) \wedge Q_m(t, \varepsilon))$
 $Q_m(t, \varepsilon) = Q(t) + \varepsilon Q^{(1)}(t) + \varepsilon^2 Q^{(2)}(t) + \dots = W_m(t, \varepsilon) Q$
D.t. $U_\varepsilon(t) Q_m(0, \varepsilon) = \exp\left(-\frac{i}{\varepsilon} \int_0^t E_m(s, \varepsilon) ds\right) Q_m(t, \varepsilon) + O_m(\varepsilon^{m+1} t).$

Goplen adiabatic Thm.

Suited to deal with situations where the eigenvalue of interest fails to be isolated in the spectrum.

Avron-Elgart '99, simplified by Teschl '01, see also Bornemann '98.

Setup: \mathcal{H} a separable Hilbert space

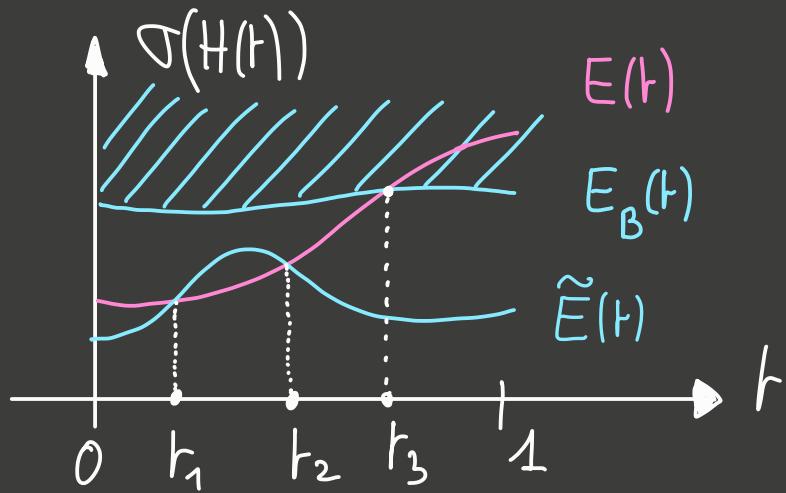
$[0,1] \ni t \mapsto H(t) = H^*(t)$; strongly C^2 , bounded (for simplicity)

$$U_\varepsilon(t) \text{ s.t. } i\varepsilon U'_\varepsilon(t) = H(t)U_\varepsilon(t); \quad U_\varepsilon(0) = \mathbb{I}.$$

Spectral assumption: $\forall t \in [0,1],$

- $E(t)$ eigenvalue of $H(t)$, $P(t)$ shr- $C^2[0,1]$ family of finite rank projectors
- $H(t)P(t) = E(t)P(t)$
- $P(t)$ is the spectral projection of $H(t)$ on $\{E(t)\}$ for almost all $t \in [0,1]$.

Example :



the spectral proj. $\prod \{E(t)\} (H(t))$
is not continuous at $\{t_1, t_2, t_3\}$.

let $V_\varepsilon(t)$ defined by $i\varepsilon V'_\varepsilon(t) = (H(t) + i\varepsilon [P'(t), P(t)]) V_\varepsilon(t)$; $V_\varepsilon(0) = I$

$$\text{D.f. } V_\varepsilon(t) P(0) = P(t) V_\varepsilon(t), \quad \forall t \in [0, 1].$$

Theorem (Aron-Elgart, Tenfel)

$$\sup_{t \in [0, 1]} \|U_\varepsilon(t) - V_\varepsilon(t)\| \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \text{so that} \quad \|(I - P(t)) U_\varepsilon(t) P(0)\| \xrightarrow{\varepsilon \rightarrow 0} 0$$

Recall: $V_\varepsilon(t) = \omega(t) \phi_\varepsilon(t)$, hence the intertwining property.

Proof: Define $S_\varepsilon(t)$ by $U_\varepsilon(t) = V_\varepsilon(t) S_\varepsilon(t)$ s.t. $S_\varepsilon(t) = \mathbb{I} + \int_0^t \bar{V}_\varepsilon^{-1} [P, P'] U_\varepsilon ds$.

In the gapped case, to integrate by parts, we solved

$$[P(s), P'(s)] = [H(s), X(s)] \quad \text{with} \quad X(s) = \frac{1}{2\pi i} \int_{\Gamma} (H(s) - z)^{-1} P'(z) (H(z) - z)^{-1} dz$$

Here we find $X(s)$ and $Y(s)$ s.t.

$$[P(s), P'(s)] = [H(s), X(s)] + Y(s) \quad (\text{partial resolution})$$

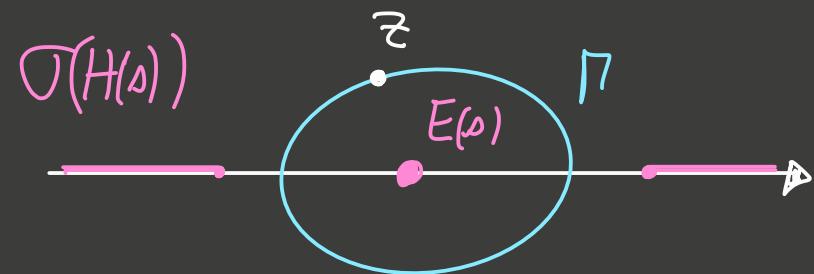
so that

$$\begin{aligned} S_\varepsilon(t) &= \mathbb{I} + \int_0^t \bar{V}_\varepsilon^{-1} ([H, X] + Y) U_\varepsilon ds \\ &= \mathbb{I} + \int_0^t \left\{ -i\varepsilon \frac{d}{ds} (\bar{V}_\varepsilon^{-1} X U_\varepsilon) + i\varepsilon \bar{V}_\varepsilon^{-1} X' U_\varepsilon + i\varepsilon \bar{V}_\varepsilon^{-1} [P', P] X U_\varepsilon + \bar{V}_\varepsilon^{-1} Y U_\varepsilon \right\} ds \end{aligned}$$

Hence:

$$\sup_{t \in [0, 1]} \|U_\varepsilon(t) - V_\varepsilon(t)\| \leq \varepsilon \sup_{s \in [0, 1]} 2 \|X(s)\| \left(1 + \|P'(s)\| \right) + \int_0^1 (\varepsilon \|X'(s)\| + \|Y(s)\|) ds$$

Gapped case:



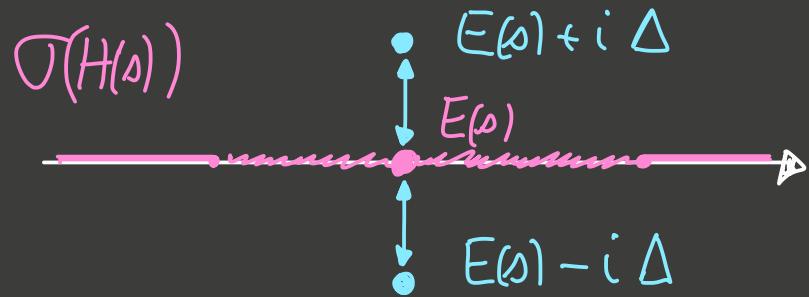
$$(H(s) - z)^{-1} = \frac{P(s)}{E(s) - z} + R(z, s)$$

$$Y_G(s) \equiv 0;$$

$$X_G(s) = \frac{1}{2\pi i} \int_{\Gamma} (H(s) - z)^{-1} P'(s) (H(s) - z)^{-1} dz = -R(E(s), s) P'(s) P(s) - P(s) P'(s) R(E(s), z)$$

No gap: shift the resolvent :

$$\text{let } \Delta > 0$$



$$X_\Delta(s) := -R(E(s) - i\Delta, s) P'(s) P(s) - P(s) P'(s) R(E(s) + i\Delta, s) \quad (= X_\Delta^*(s))$$

$$Y_\Delta(s) := +i\Delta R(E(s) - i\Delta, s) P'(s) P(s) + i\Delta P(s) P'(s) R(E(s) + i\Delta, s) \quad (= -Y_\Delta^*(s))$$

Lemma:

$$[P(s), P'(s)] = [H(s), X_\Delta(s)] + Y_\Delta(s)$$

(use $H(s)P(s) = E(s)P(s)$, $H(s)R(z, s) = (H(s) - z)R(z, s) + zR(s, z) = (\mathbb{I} - P(s)) + zR(s, z)$)

Find a priori bounds: $\exists C < \infty$ s.t.

$$\sup_{\rho \in [0,1]} \|X_\Delta(\rho)\| \leq \frac{C}{\Delta} ; \quad \sup_{\rho \in [0,1]} \|Y_\Delta(\rho)\| \leq C ; \quad \sup_{\rho \in [0,1]} \|X'_\Delta(\rho)\| \leq \frac{C}{\Delta^2}$$

$$\left(\text{using } \|R(z,s)\| \leq \frac{1}{|\Im z|} ; \quad \|\partial_\rho R(z,\rho)\| \leq \frac{C}{|\Im z|^2} ; \quad E(\rho) = \frac{k_2(H(\rho)P(\rho))}{k_2(P(\rho))} \in C^2 \right)$$

Hence, for some $C' < \infty$

$$\sup_{t \in [0,1]} \|U_\varepsilon(t) - V_\varepsilon(t)\| \leq C' \left(\frac{\varepsilon}{\Delta} + \frac{\varepsilon}{\Delta^2} + \int_0^1 \|Y_\Delta(s)\| ds \right).$$

With the following lemma,

Lemma: $\boxed{\|Y_\Delta(s)\| \xrightarrow{\Delta \rightarrow 0} 0, \text{ almost all } \rho \in [0,1]}$,

$\Delta(\varepsilon)$ s.t. $\Delta(\varepsilon) \& \frac{\varepsilon}{\Delta(\varepsilon)} \xrightarrow{\varepsilon \rightarrow 0} 0$, and dominated convergence yield the result.

Proof of the lemma:

i) Enough to look at $Y_{\Delta}(\omega) P(\omega) = i \Delta R(E(\omega) - i\Delta, \omega) P'(\omega) P(\omega)$ (adjoint)

ii) $P(\omega)$ finite rank \Rightarrow enough to prove $\lim_{\Delta \rightarrow 0} Y_{\Delta}(\omega) P(\omega) \psi = 0, \forall \psi \in \mathcal{H}$.

iii) $\varphi(\omega) := P'(\omega) P(\omega) \psi \Rightarrow \varphi(\omega) \in \text{Ran}(I - P(\omega))$ (PP'P = 0)

$$\Rightarrow \|Y_{\Delta}(\omega) P(\omega) \psi\|^2 = \|\Delta R(E(\omega) - i\Delta) \varphi(\omega)\|^2 = \Delta^2 \|\overline{(H(\omega) - (E(\omega) - i\Delta))^{-1}} \varphi(\omega)\|^2$$

$$= \int_{\sigma(H(\omega))} d\mu_{\varphi(\omega)}(\lambda) \frac{\Delta^2}{(\lambda - E(\omega))^2 + \Delta^2} \xrightarrow{\Delta \rightarrow 0} \mu_{\varphi(\omega)}(\{E(\omega)\})$$

spectral measure
of $H(\omega)$ for $\varphi(\omega)$

But $\varphi(\omega) \in \text{Ran}(I - P(\omega)) \Rightarrow \mu_{\varphi(\omega)}(\{E(\omega)\}) = 0, \text{ a.a. } \omega \in [0, 1].$ □

Remarks:

- If one has information on $d\eta_{Q(0)}$, one may get a rate of decay.
- Proving that $P \in C^2([0, 1])$ can be difficult.
- Application: ground state at the bottom of a.c. spectrum.
- Precursors:
 - Bon-Fode '28 finitely many crossings of e.v.
 - Hagedorn '85 non generic crossings
 - Aron Howland Simon '90 dense point spectrum
 - Gérard Jauslin J. Monthi '99 infinitely many crossings on a compact
 - Ergodic - Hagedorn '10 adiabatic limit for resonances

⋮