

Grenoble
Quantum Meeting
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Adiabatic quantum dynamics and applications

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(Tentative) Program :

1) Adiabatic Theorems in Quantum Mechanics

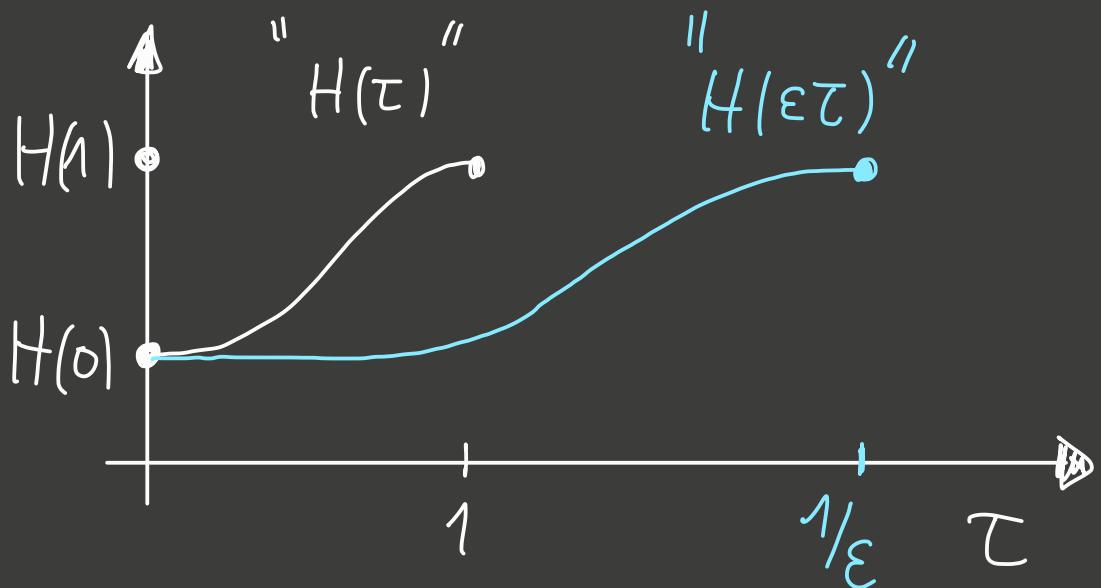
- "à la Kato"
- superadiabatic
- exponential
- gapless

2) Applications to open quantum systems

- Lindbladians
- dephasing Lindbladians
- Two-level system in a Box field

Adiabatic Theorems in Quantum Mechanics

Context: approximate solution to the time-dependent Schrödinger equation with slowly varying time-dependent Hamiltonian



$H: [0, 1] \rightarrow \mathcal{L}(\mathcal{H})$, \mathcal{H} separable Hilbert space
 $t \mapsto H(t) = H^*(t)$; smooth

Determine $X_\varepsilon(t) \in \mathcal{H}$ for $t \in [0, 1/\varepsilon]$ s.t.

$$\begin{cases} i \partial_t X_\varepsilon = H(\varepsilon t) X_\varepsilon ; & t \in (0, 1/\varepsilon) \\ X_\varepsilon(0) = \varphi_\varepsilon ; & \text{as } \varepsilon \rightarrow 0 . \end{cases}$$

Change of variable : $t := \varepsilon \tau \in [0, 1]$; $\Psi_\varepsilon(t) := X_\varepsilon(t/\varepsilon)$

$$\begin{cases} i \varepsilon \partial_t \Psi_\varepsilon = H(t) \Psi_\varepsilon ; & t \in (0, 1) \\ \Psi_\varepsilon(0) = \varphi_\varepsilon ; & \text{as } \varepsilon \rightarrow 0 . \end{cases}$$

Determine the propagator $U_\varepsilon: [0, 1] \times [0, 1] \rightarrow \mathcal{L}(\mathcal{H})$

$$\text{D.t. } \begin{cases} i \varepsilon \partial_t U_\varepsilon(t, t_0) = H(t) U_\varepsilon(t, t_0) \\ U_\varepsilon(t_0, t_0) = \mathbb{I} ; & \text{as } \varepsilon \rightarrow 0 . \end{cases}$$

Remark:

For $H(t) \in C^2([0,1]; \mathcal{L}(\mathcal{H}))$, existence and uniqueness thanks to Dyan series.

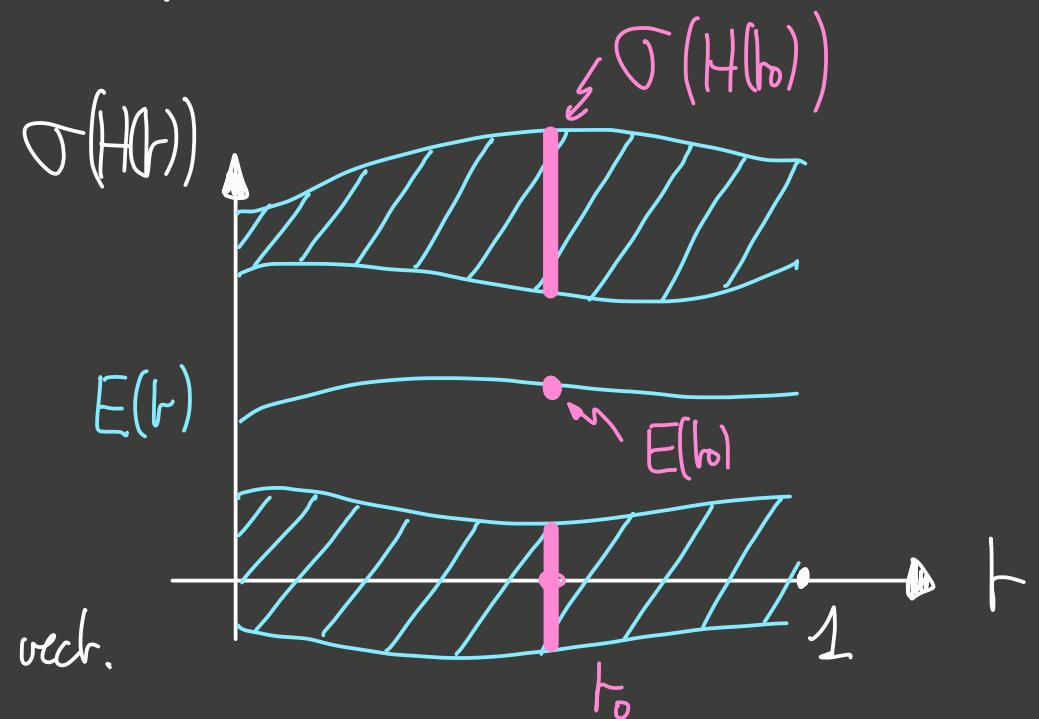
Slogan: "Once in an eigenstate, always in an eigenstate":

Let $H(t) = H^*(t)$ o.b.

$E(t) \in \sigma(H(t))$ isolated simple eigenvalue

$$\varphi(t) \in \mathcal{H} \quad H(t)\varphi(t) = E(t)\varphi(t); \quad \|\varphi(t)\| = 1$$

and $\langle \varphi(t) | \varphi'(t) \rangle = 0$, instantaneous eigen vec.



$$\text{Let } t_0 \in [0,1], \quad \Psi_\varepsilon(t, t_0) := U_\varepsilon(t, t_0) \varphi(t_0) \quad \text{o.b.} \quad \begin{cases} i\varepsilon \partial_t \Psi_\varepsilon(t, t_0) = H(t) \Psi_\varepsilon(t, t_0) \\ \Psi_\varepsilon(t_0, t_0) = \varphi(t_0) \end{cases}$$

Then,

$$\boxed{\Psi_\varepsilon(t, t_0) = e^{-\frac{i}{\varepsilon} \int_{t_0}^t E(s) ds} \varphi(t) + O(\varepsilon) \quad \forall t, t_0 \in [0,1]} \quad \begin{matrix} \text{Bam-Ford '28} \\ \text{Kato '50} \end{matrix}$$

Note: We will often take $t_0 = 0$ and write $U_\varepsilon(t) := U_\varepsilon(t, 0)$, s.t. $U_\varepsilon(0) = I$.

Upgrade: Let \mathcal{H} be a separable Hilbert space. Assume:

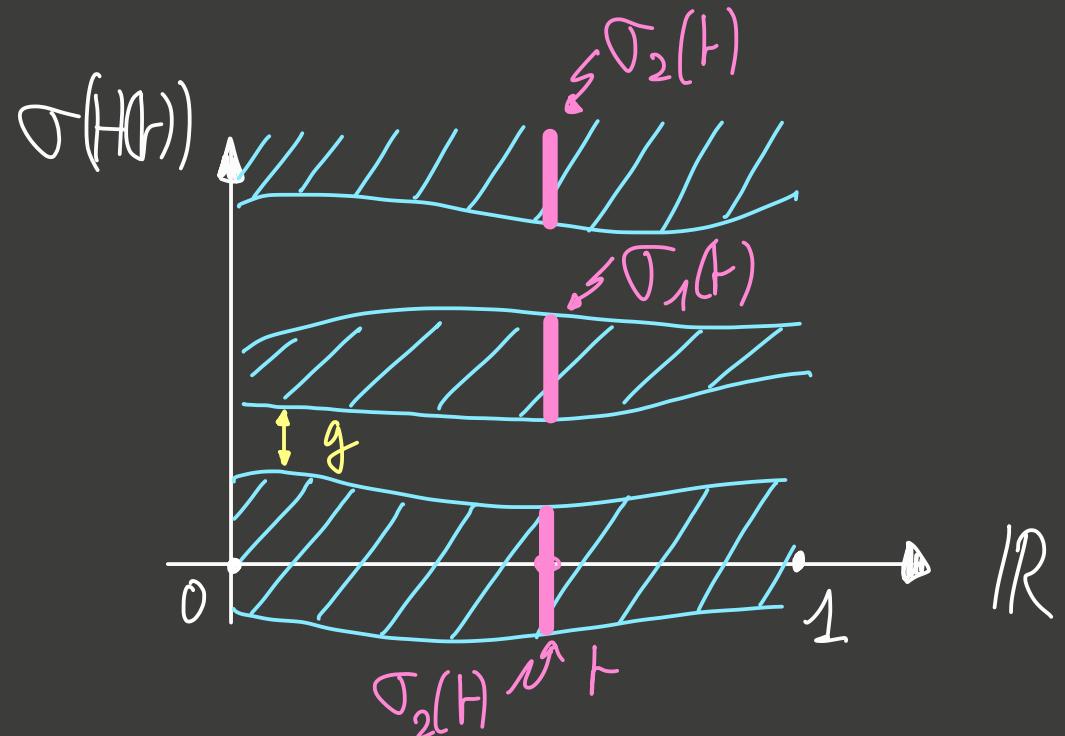
(R) $[0,1] \ni t \mapsto H(t) = H^*(t) \in \mathcal{L}(\mathcal{H})$ (σ defined on $\mathcal{D} \subset \mathcal{H}$ indep. of t)

H strongly C^2 on $[0,1]$; $\exists b \in \mathbb{R}$ s.t. $H(t) \geq b$, $\forall t \in [0,1]$.

(F) $\sigma(H(t)) = \sigma_1(t) \cup \sigma_2(t)$ s.t.

$\sigma_1(t)$ bounded, $\exists g > 0$ s.t.

$$\inf_{t \in [0,1]} \text{dist}(\sigma_1(t), \sigma_2(t)) \geq g > 0$$



Riesz Projection:

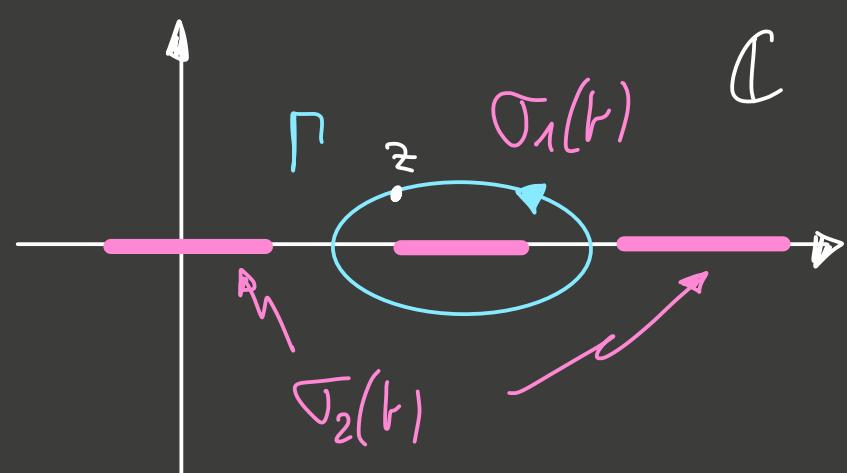
$$\text{Let } P(t) := -\frac{1}{2\pi i} \int_{\Gamma} (H(t) - z)^{-1} dz \longleftrightarrow \sigma_1(t)$$

$$P^\perp(t) := I - P(t) \longleftrightarrow \sigma_2(t)$$

$$\text{s.t. } P(t) = P^2(t) = P^*(t); \quad t \mapsto P(t) \text{ is } C^2([0,1])$$

$$[H(t), P(t)] \equiv 0; \quad (P(t): \mathcal{H} \rightarrow \mathcal{D})$$

$$\sigma(H(t)|_{P(t)\mathcal{H}}) = \sigma_1(t), \quad \sigma(H(t)|_{P^\perp(t)\mathcal{H}}) = \sigma_2(t).$$



Intertwining or // - transport or Kato operator:

Let $[0,1] \ni t \rightarrow P(t) = P^2(t) \in \mathcal{L}(\mathcal{B})$; \mathcal{B} a Banach, be strongly C^1

Consider $W(t) \in \mathcal{L}(\mathcal{B})$ the solution to $\begin{cases} W'(t) = [P'(t), P(t)] W(t) \\ W(0) = \mathbb{I} \end{cases}$ (with $' \Rightarrow \partial_t$)

Lemma: $W(t) P(0) = P(t) W(t)$, $\forall t \in [0,1]$. Kato '50

Rem: On a Hilbert, if $P(t) = P^*(t) \Rightarrow W(t)$ unitary

In any case $\tilde{W}'(t) \in \mathcal{L}(\mathcal{B})$ exists and $\begin{cases} \tilde{W}'(t)' = -\tilde{W}'(t) [P'(t), P(t)] \\ \tilde{W}'(0) = \mathbb{I}. \end{cases}$

If $\varphi = P(0)\varphi \Rightarrow W(t)\varphi = P(t)W(t)\varphi$.

On a Hilbert, if $P(t) = P^*(t)$ has rank 1 $\Rightarrow P(0) = |\varphi\rangle\langle\varphi|$ and $\varphi(t) := W(t)\varphi$

$\Rightarrow \|\varphi(t)\| = 1$; $\langle\varphi(t)|\varphi'(t)\rangle = 0$ and $P(t) = |\varphi(t)\rangle\langle\varphi(t)|$

Proof: $w(t)P(0)$ and $P(t)w(t)$ satisfy the same linear diff. eq. with initial condition $P(0)$:
 (Writing $A \equiv A(r)$; $A(0) = A(0)$)

- $P^2 = P \Rightarrow P'P + PP' = P' \Rightarrow PP'P = 0.$

- $(wP(0))' = [P', P]wP(0)$

- $(PW)' = P(P'P - PP')W + P'W = -PP'W + (P'P + PP')W = P'P(PW) = [P', P](PW)$

Finally: if $P = P^* = P^2$, and $\varphi_0 = P(0)\psi_0$, for $\varphi(t) := w(t)\psi_0$ we have

$$\langle \varphi | \varphi' \rangle = \langle P\varphi | [P', P]\varphi \rangle = \langle \varphi | P[P', P]P\varphi \rangle \equiv 0.$$

□

Dynamical phase operatr: ϕ_ε bdd in \mathcal{B} , a Banach space, defined by

$$i\varepsilon \phi'_\varepsilon(t) = \begin{pmatrix} W(t) & H(t) & W(t) \end{pmatrix} \phi_\varepsilon(t) \quad \& \quad \phi_\varepsilon(0) = \mathbb{I}, \quad \text{where } [H(t), P(t)] \equiv 0.$$

Lemma: $[\phi_\varepsilon(t), P(0)] = 0 \quad \forall t \in [0,1].$ (Evolution within $P(0)\mathcal{B}$.)

Proof: $P(0)$ $\phi_\varepsilon(t)$ and $\phi_\varepsilon(t) P(0)$ satisfy the same lin. diff. eq. with initial cond. $P(0).$

Rem: In a Hilbert space with $H(t) = H^\dagger(t)$ $\phi_\varepsilon(t)$ is unitary.

If, moreover, $P(t) \leftrightarrow \sigma_1(t) := \{E(t)\} \Rightarrow \phi_\varepsilon(t)|_{P(0)\mathcal{H}} = e^{-\frac{i}{\varepsilon} \int_0^t E(s) ds} P(0).$

Adiabatic evolution:

We now define $V_\varepsilon(t) := W(t) \phi_\varepsilon(t).$ Then $V_\varepsilon(t) P(0) = P(t) V_\varepsilon(t).$

Moreover: $i\varepsilon V'_\varepsilon(t) = (H(t) + i\varepsilon [P'(t), P(t)]) V_\varepsilon(t) \quad \& \quad V_\varepsilon(0) = \mathbb{I}.$

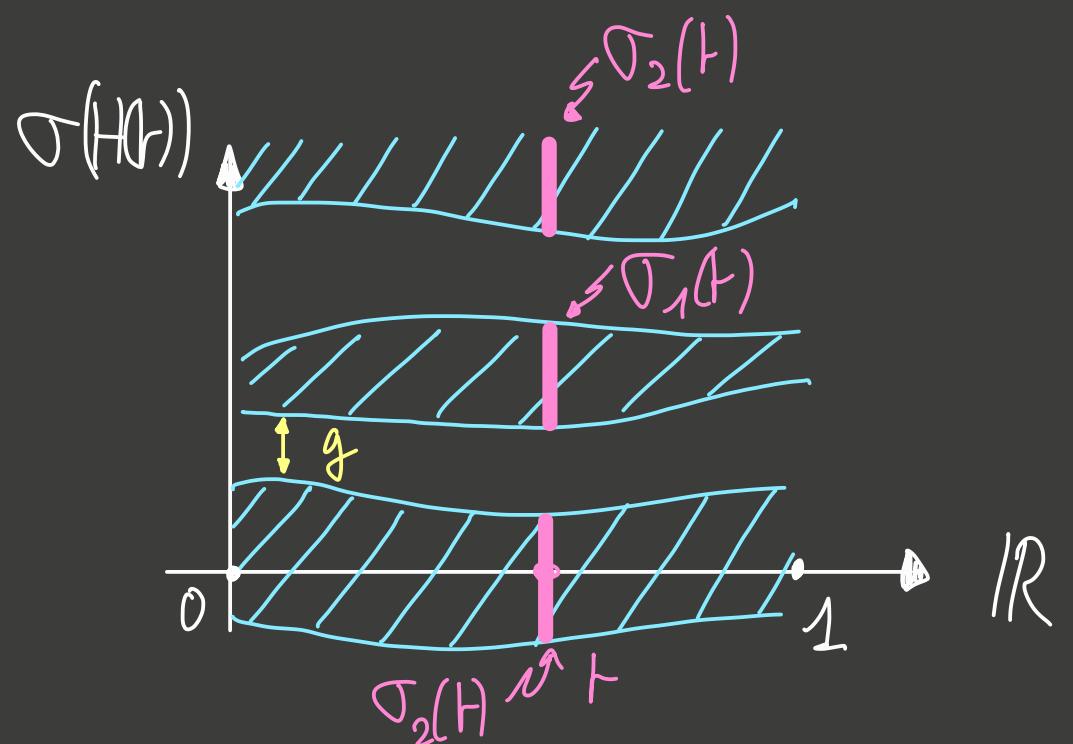
Reminder: \mathcal{H} is a separable Hilbert space.

(R) $[0,1] \ni t \mapsto H(t) = H^*(t) \in \mathcal{L}(\mathcal{H})$ (α defined on \mathcal{DCH} indep. of t)
 H strongly C^2 on $[0,1]$; $\exists b \in \mathbb{R}$ s.t. $H(t) \geq b$, $\forall t \in [0,1]$.

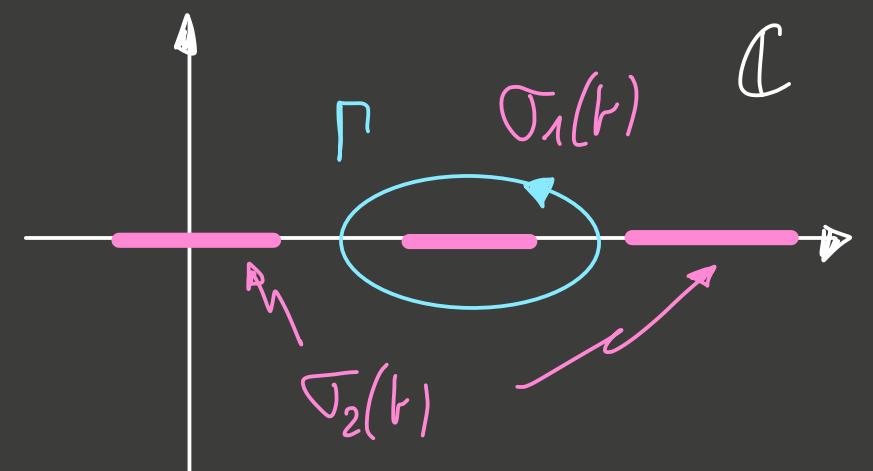
(F) $\sigma(H(t)) = \sigma_1(t) \cup \sigma_2(t)$ s.t.

$\sigma_1(t)$ bounded, $\exists g > 0$ s.t.

$$\inf_{t \in [0,1]} \text{dist}(\sigma_1(t), \sigma_2(t)) \geq g > 0$$



Let $P(t) := -\frac{1}{2i\pi} \int_{\Gamma} (H(t)-z)^{-1} dz \leftrightarrow \sigma_1(t)$



A diabatic Theorem : Assume the above. (Neucin '80, Avron - Seiler - Yaffe '87)

Let $U_\varepsilon(t)$ s.t. $i\varepsilon U'_\varepsilon(t) = H(t)U_\varepsilon(t)$ & $U_\varepsilon(0) = \mathbb{I}$, and $V_\varepsilon(t) = W(t)\phi_\varepsilon(t)$.

Then : $\exists C > 0$ s.t. $\forall t \in [0, 1]$, $\forall \varepsilon > 0$,

$$\|U_\varepsilon(t) - V_\varepsilon(t)\| \leq C(\varepsilon t) . \quad (\text{with } U_\varepsilon(t), W(t), \phi_\varepsilon(t) \text{ unitary})$$

Remarks: • If $H(t)$ defined on $I \supset [0, 1]$, the estimate above holds unif. in I .

• The transition amplitudes between spectral subspaces vanish :

$$\|(I - P(t))U_\varepsilon(t)P(0)\| = O(\varepsilon t) \Rightarrow \text{transition probabilities are of order } \varepsilon^2. \quad (0 \leq t \leq 1)$$

• Reduces to the case of Kato in case $\text{rank } P(t) = 1$ & $\Sigma_1(t) = \{E(t)\}$

• Estimate on the transition amplitudes between spectral subspaces of H is sharp.

• Estimate on the evolution op. can (and will) be improved.

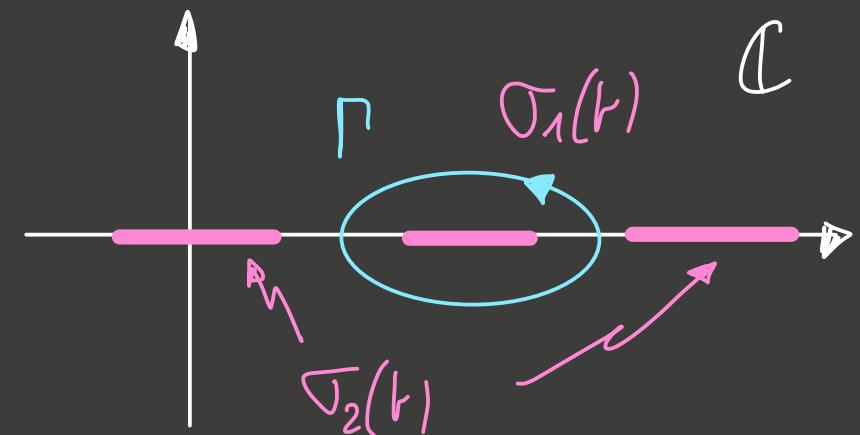
Proof: For short, write A for $A(H)$, $A(0)$ for $A(0)$.

$$\text{Define } \mathcal{L}_\varepsilon(t) \text{ by } U_\varepsilon(t) = V_\varepsilon(t) \mathcal{L}_\varepsilon(t) \iff \mathcal{L}_\varepsilon(t) = \tilde{V}_\varepsilon(t) U_\varepsilon(t).$$

$$i\varepsilon \mathcal{L}'_\varepsilon = -\tilde{V}_\varepsilon^{-1} (H + i\varepsilon [P', P]) U_\varepsilon + \tilde{V}_\varepsilon^{-1} H U_\varepsilon = -i\varepsilon \tilde{V}_\varepsilon^{-1} [P', P] U_\varepsilon; \quad \mathcal{L}_\varepsilon(0) = I.$$

$$\text{i.e.: } \mathcal{L}_\varepsilon(t) = I + \int_0^t \tilde{V}_\varepsilon^{-1} [P, P'] U_\varepsilon ds$$

Integration by parts: Let $B \in \mathcal{L}(H)$ and set



$$\mathcal{R}(B) := -\frac{1}{2i\pi} \int_R (H-z)^{-1} B (H-z)^{-1} dz : \mathcal{H} \rightarrow \mathcal{D}. \quad \text{Then: } [\mathcal{R}(B), H] = [P, B].$$

$$\text{Indeed: } -\frac{1}{2i\pi} \int_R [(H-z)^{-1} B (H-z)^{-1}, H] dz \stackrel{H \mapsto H-z}{=} -\frac{1}{2i\pi} \int_R [(-z)^{-1}, B] dz.$$

For $B = P'$, $[\mathcal{R}(P'), H] = [P, P']$, where $s \mapsto \mathcal{R}(P')(s) \in \mathfrak{sh}-C^1$.

$$\begin{aligned} \Rightarrow \mathcal{L}_\varepsilon(t) - I &= \int_0^t \tilde{V}_\varepsilon^{-1} [\mathcal{R}(P'), H] U_\varepsilon ds = \int_0^t i\varepsilon \frac{d}{ds} \left\{ \tilde{V}_\varepsilon^{-1} \mathcal{R}(P') U_\varepsilon \right\} ds \\ &\quad + \int_0^t \left\{ i\varepsilon \tilde{V}_\varepsilon^{-1} [P', P] \mathcal{R}(P') U_\varepsilon - i\varepsilon \tilde{V}_\varepsilon^{-1} (\mathcal{R}(P'))' U_\varepsilon \right\} ds = O(\varepsilon t). \quad \square \end{aligned}$$

unitarity

Remarks

- Besides regularity in t , existence of the propagators involved, separation of $\sigma(H(t))$, no need for self adjointness of $H(t)$ to get the integral expression for $S_\varepsilon(t) - \mathbb{I}$.
- To conclude, it is enough to have $\sup_{\substack{t \in [0,1] \\ \varepsilon > 0}} \|V_\varepsilon^{\pm 1}(t)\| = v < \infty \Leftrightarrow \sup_{\substack{t \in [0,1] \\ \varepsilon > 0}} \|\phi_\varepsilon^{\pm 1}(t)\| < \infty$

$$\text{Indeed } \|S_\varepsilon(t) - \mathbb{I}\| \leq c \cdot v \varepsilon \sup_{t \in [0,1]} \|U_\varepsilon(t)\|$$

$$\Rightarrow \|U_\varepsilon(t)\| = \|V_\varepsilon(t) S_\varepsilon(t)\| \leq v + c v^2 \varepsilon \sup_{s \in [0,1]} \|U_\varepsilon(s)\|$$

$$\text{i.e. } \sup_{s \in [0,1]} \|U_\varepsilon(s)\| \left(1 - c v^2 \varepsilon\right) \leq v \Rightarrow \sup_{s \in [0,1]} \|U_\varepsilon(s)\| \leq 2v \quad \forall \varepsilon \leq \frac{1}{2c v^2}.$$

$$\text{and eventually } S_\varepsilon(t) - \mathbb{I} = O(\varepsilon).$$